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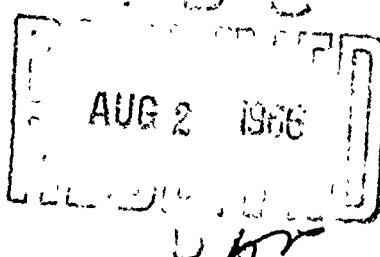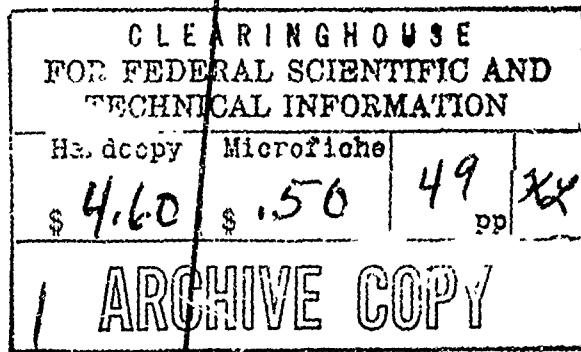APPLIED PHYSICS LABORATORY
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WAVE LENGTH LENSES

by
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Gilbert Wilkes

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Page

10   Equation (29) should read:

$$R(d)^2_{REAL} = \left(\frac{1}{a_1^2 + a_2^2}\right)^2 (a_1^2 - a_2^2)\left(K'^2 - e^{-2md}(b_2^2 - b_1^2) - 2K'b_1 e^{-md}\right)$$

12   Lines 6 and 7 should read:

The time for the signal to reach the position defined by l_1' will be $t_1' = \frac{d}{c} + \frac{l_1'}{v}$ and that for the position defined by l_2' will be $t_2' = \frac{d}{c} + \frac{l_2'}{v}$

19   Equation (54) should read:

$$F_2 = \frac{\sin\left(\frac{\pi p}{n_L} \sin^2 \theta\right)}{\sin\left(\frac{\pi}{n_L} \sin^2 \theta\right)}$$

19   Equation (56) should read:

$$F_2 = \frac{\sin\left(\frac{\pi p}{n_L} \sin \theta\right)}{\sin\left(\frac{\pi}{n_L} \sin \theta\right)}$$

24   Equation (68) should read:

$$F_3 = \frac{\sin n_3 \frac{\gamma_3}{2}}{\sin \frac{\gamma_3}{2}}$$

24   Last line on page should read:

$$\frac{2\pi a_3}{\lambda} \cos \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{K}{2}\pi.$$

K being odd integer, $\cos \theta_{1/2} = \frac{\lambda K}{4a_3}$

WAVE LENGTH LENSES[1]

Introduction

The property shown by dielectric blocks of concentrating the energy of electromagnetic waves forms the subject of this report. This property is similar in many respects to the operation of an ordinary glass optical lens, and for this reason blocks of dielectric designed to concentrate radiant electromagnetic energy are called lenses, although in appearance they do not resemble optical lenses.

The energies considered in the following have a much lower frequency than that of light, and the wave lengths are generally such that the lenses have dimensions of the order of a wave length, for the more usual applications. In this they differ radically from optical lenses. A definite phase relationship is found to exist between the energy in the lens and that of free space, and if in the design of the lens, this relationship is not respected, the lens will not yield its maximum concentrating power, or gain. The energy velocity in a lens approaches that of light with a decreasing cross section. The side walls of a lens are found to be effective energy gatherers. The thinner the lens, the longer it can be made so as to increase its exposed area. An increased gain results. The index of refraction of the lens material plays a role in this action by its effect on the lens velocity.

Experimental data correlating their different properties is given in the following with tentative supporting theories where possible.

Dielectric losses contribute a small and usually negligible effect in the operation of all lenses, but this effect has been disregarded to simplify as far as possible the following data.

Note:[1] The term "wave length lens" is used in preference to "polyrod antenna" because in my opinion the former expresses the lenslike concentration of energy of the device, i.e., its similarity to optical lenses.

The wave length lens dimensions can be caused to vary continuously in any direction and the resulting variations of its characteristics are smooth and continuous as is obtained in optical lenses. The term antenna calls to my mind a structure that can only vary by integers such as an end fire array. We cannot speak of a two-and-a-half oscillator antenna, whereas a lens two-and-a-half wave lengths long is quite reasonable.

GAIN

If a receiver of electromagnetic energy is equipped with a properly designed block of dielectric or lens, the signal from the receiver is found to be greater than that obtainable with the bare receiver. The ratio of the signal from the lens equipped receiver to that of the bare receiver is defined as the relative gain of the lens. It is the relative increase in energy received when the lens is in place to that received without the lens. Conversely the same gain of energy is obtained at a distant receiver by an emitter equipped with a lens compared to the emitter without the lens.

The gain on axis of a lens is the characteristic of principal interest and unfortunately the one least reducible to a simple quantitative expression. It is the result not only of the lens material and dimensions, but of the receiver to which the lens is coupled, and of interactions between the receiver and lens.

For example, if a series of blocks of constant cross section but varying length are set in front of a horn or wave guide of the same cross section, the gain is found to vary periodically with length. One variation of small amplitude and short period is generally discernible, but there is a longer high amplitude variation that is most striking, as shown on all the attached curves of gain and particularly in Curve No. 1.

The first variation is found to correspond to Snell's law of reflection from a thin sheet with normal incidence. The well known formula for the transmitted flux for a sheet thickness, "d" under these conditions is:

$$T = 1-R = 1 - \frac{(r_{12} + r_{23})^2 - r_{12}\, r_{23} \sin^2 \alpha d}{(1 + r_{12}\, r_{23})^2 - 4 r_{12}\, r_{23} \sin^2 \alpha d} \qquad (1)$$

in which:

$$r_{jk} = \frac{\sqrt{\varepsilon_j} - \sqrt{\varepsilon_k}}{\sqrt{\varepsilon_j} + \sqrt{\varepsilon_k}} \qquad (2)$$

ε_j = Dielectric constant of material j

ε_k = Dielectric constant of material k

$\alpha = \dfrac{2\pi}{\lambda_L}$

λ_L = Wave length in sheet or lens.

If end effects are disregarded, equation (2) gives: $r_{12} = -r_{21}$ (3) and equation (1) passes through minima for:

$$\sin \alpha d = 1 \text{ or } \frac{2 \pi d}{\lambda_L} = K \frac{\pi}{2}, K = 1, 3, 5 \qquad (4)$$

which can then give a direct measurement of $\dot{\lambda}_L$. This proved of considerable value in checking the apparent index of refraction in lenses discussed later on. Otherwise, the undesirable effect of this reflection on gain is slight and can be reduced or eliminated by making the lens pointed.

The energy of a lens increases with its length up to a certain point after which it falls to a value corresponding approximately to the gain of the bare mouth, after which it increases again. The side walls can be shown to be responsible for this behavior by covering them with a resistive material which destroys the gain of the lens. The mechanism of this gathering action of the side walls seems to have to do with the internal angle of total reflection in the dielectric. The dielectric is probably traversed by internal displacement currents that set up their own radiation in the dielectric, some of which is totally reflected or entrapped within the lens and is not returned to the external field.

This energy is found to travel inside the dielectric and is not confined to the surface. For example, a very small hollow metallic tube capable of transmitting the wave length under consideration can have its open end imbedded in a much larger dielectric block, and the energy in the tube will be found to be increased by the concentrating action of the block.

A dielectric block is found to have no lower size limit or cut-off dimension when exposed to an external field. It differs in this from a dielectric filled wave guide, in which the cut-off dimension is:

$$\lambda g = \frac{\lambda}{\sqrt{\mathcal{E} - \left(\frac{\lambda}{\lambda c} \right)^2}} \qquad (5)$$

in which

λ_g = Wave length in dielectric filled metallic guide

λ = Wave length in free space

$\mathcal{E}$ = Dielectric Constant

λc = Cut-off wave length of hollow guide

λ_c = 2b for gravest mode rectangular guide

d = Dimension of wave guide normal to E vector

For energy to be transmitted along the guide, $\lambda'g$ must be real and finite or

$$\varepsilon = \left(\frac{\lambda}{\lambda c}\right)^2 > 0 \tag{6}$$

which leads to

$$2b\sqrt{\varepsilon} > \lambda \tag{7}$$

If this is not obtained, transmission in a wave guide is impossible. On the other hand, if the sides of the dielectric wave guide are exposed to the external field, we have a lens and find the above limitation does not exist; in fact, the more powerful lenses are below this limit.

The absence of cut-off under these conditions is discussed by Schelkunoff[2] whose data is summarized as follows:

> "If a dielectric rod is subjected to a wave in a non-dissipative medium, the waves will be circularly symmetric and hybrid. They will have E and H components parallel to the rod - in the rod (for gravest mode).
>
> $$E_z = AJ_1(\chi\rho)\cos\phi \qquad H_z = BJ_1(\chi\rho)\sin\phi \tag{8}$$
>
> and in the medium
>
> $$E_z = CK_1(k\rho)\cos\phi \qquad H_z = DK_1(k\rho)\sin\phi \tag{9}$$

in which the propagation factor $e^{-\Gamma z + \iota\omega t}$ is implied. The transverse phase constant χ is such that for the rod

$$\chi^2 = \Gamma^2 + \beta_1^2 \tag{10}$$

---

[2]Schelkunoff. "Electro Magnetic Waves" Pages 425-428.

for the medium

$$k^2 = -\Gamma^2 - \beta_2^2 \tag{11}$$

(transverse propagation constant)

Schelkunoff defines β_1 and β_2 as the intrinsic phase constants in the rod and medium as:

$$\beta_1 = \omega\sqrt{\mu_1 \varepsilon_1} \qquad \beta_2 = \omega\sqrt{\mu_2 \varepsilon_2} \tag{12}$$

The ϕ components of the field are given (for the rod) as

$$E\phi = \left[A\frac{\Gamma}{\chi^2\rho} J_1(\chi\rho) + B\frac{\iota\omega\mu_1}{\chi} J_1'(\chi\rho) \right] \sin\phi \tag{13}$$

$$H\phi = \left[A\frac{\iota\omega\varepsilon_1}{\chi} J_1'(k\rho) + B\frac{\Gamma}{\chi^2\rho} J_1(\chi\rho) \right] \cos\phi \tag{14}$$

and for the external medium:

$$E\phi = -\left[C\frac{\Gamma}{k^2\rho} K_1(k\rho) + D\frac{\iota\omega\mu_2}{k} K_1'(k\rho) \right] \sin\phi \tag{15}$$

$$H\phi = \left[C\frac{\iota\omega\varepsilon_2}{k} K_1'(k\rho) + \frac{\Gamma}{k^2\rho} K_1(k\rho) \right] \cos\phi \tag{16}$$

If the rod has a radius "a" the tangential intensities of the internal and external fields are continuous for ρ = a."

After simplifying, the result is:

$$\frac{-\mu_1\varepsilon_1 J_0(\chi_a) J_2(\chi_a)}{\chi^2 a^2 J_1^2(\chi_a)} + \frac{(\mu_1\varepsilon_2 + \mu_2\varepsilon_1) J_1'(\chi_a) K_1'(ka)}{\chi a k a J_1(\chi_a) K_1(ka)} \tag{17}$$

$$\frac{+\mu_2\varepsilon_2 K_0(ka) K_2(ka)}{K^2 a^2 K_1^2(ka)} = \frac{\mu_1\varepsilon_1 + \mu_2\varepsilon_2}{\chi^2 a^2 k^2 a^2}$$

Schelkunoff goes on to prove that if k_a is made zero, χa must become a root of J_1 or $\chi a \longrightarrow 0$ also. There is no lower limit of "a" for which transmission does not occur.

However, if $k \longrightarrow 0$ then $\Gamma = \iota \beta_2$ or:

$$\Gamma = \iota 2\pi \nu \sqrt{\mu_2 \varepsilon_2} = \frac{\iota 2\pi \nu}{c} = \frac{\iota 2\pi}{\lambda} \tag{18}$$

(for air) and the propagation function for free space becomes:

$$e^{\Gamma'z - \iota \omega t} = e^{\frac{\iota 2\pi z}{\lambda} - \iota 2\pi \nu t} \tag{19}$$

and has a maximum for

$$\frac{2\pi z}{\lambda} - 2\pi \nu t = 0, \pi, 2\pi \ldots \text{ or} \tag{20}$$

$$\frac{z}{\lambda} = \nu t_0$$

The time required for the energy to traverse one centimeter at the speed of light ($\lambda \nu = c$) is therefore t_0, which is the function of an undisturbed plane wave. But putting $k = 0$ to verify the above equation requires that χa approach a root of J_1. The first of these is $\chi a = 0$ and, a or $\chi = 0$. If $\varepsilon_1 \neq \varepsilon_0 \chi \neq 0$ so "a" must be zero, or the rod nonexistent. As soon as the rod diameter departs from zero, the phase of the external wave is retarded in the vicinity of the rod. The equiphase surfaces cease to be plane but are inclined along the rod. The phase of a plane wave in the neighborhood of a dielectric rod travels at less than the speed of light. The bending of the phase surfaces is sometimes expressed by saying that the Poynting vector becomes bent into the rod.

Schelkunoff observes that the field varies exponentially for large values of $k\rho$. The field is concentrated toward the center of the rod.

An attempt to formulate the overall phase velocity in the rod by means of these equations leads to considerable difficulty. After some further simplification an expression of $\chi \rho$ in function of $K_0 (k\rho)$ is obtained and as $\rho \longrightarrow 0$ this function becomes exponential. It can be expected, therefore, that for $\frac{a}{\lambda} < 1$ the phase velocity will be an inverse exponential function of "a" approaching asymptotically the velocity of light for a $\longrightarrow 0$. The results of such an analysis would not reflect the experimentally significant effects of the rod ends, and therefore do not seem of much practical value. The phase velocity in a dielectric lens is always less than that of the external field. The field in the lens at the attacking face is in time quadrature leading the external field and will gradually lose this angle of lead as the

fields advance along the lens until a lagging phase angle of 90° is reached, at which point the gain of the lens starts to decrease. Between these two extremes there always exists a region where the internal and external fluxes are in phase and maximum coupling exists. The rate of lens energy increase in this region is maximum.

If the energy phase is such as has just been described, it should be possible to secure additional gain, for a given cross section lens, by shielding those parts of the length that are out of phase with the external field, and trial shows this to be approximately true. If a long dielectric rod is placed in the direction of propagation of a wave, its relative gain can be increased by placing thin metal jackets over it designed to shield the dielectric over such a length that a lag of one-half period occurs throughout it. These jackets are then spaced so that their attacking edges are at the points where the external and internal fluxes just reach time quadrature. For each jacket so placed the relative gain increases by a nearly equal amount. As the rod gets very long the loss in the dielectric tends to decrease the increment gains. A way of avoiding the excessive dielectric losses is to sectionalize the dielectric rod so that the metal jackets are hollow. They then advance the internal phase to bring it back into the proper phase relationship. The dielectric following each section of hollow metallic wave guide is in proper phase with the external flux. Both of these arrangements confirm the conception of phase relationship between the internal and external fields, but do not appear of practical importance otherwise. It will be shown that the lens of Fig. 2 has a phase velocity almost equal to that of light. Constructive coupling exists in it over a long length (actually 5.9λ) so that a nearly uniform increase in gain up to this length should be expected. The sharp variations in gain of short period are due to the matching and mismatching effect of the dielectric in the wave guide, which is capable of transforming the wave guide impedance to that of the lens-atmosphere system. When no match is obtained, the lens transformer combination shows gains that lie nicely on the expected gain curve. When good matching is obtained the energy should be twice that obtained with no matching, which is verified.

That the above variation is due to the action of the dielectric in the wave guide may be demonstrated by leaving a portion of the dielectric fixed in the guide and extending the lens gradually outwards as shown on Fig. 3. No matching exists here so that this curve falls on the minima of the preceding one.

Now to get back to the actual gains of a lens. The phase velocity of a lens, having a uniform cross section and a known apparent index of refraction, will be $v_L = \frac{c}{n_L}$. c being the velocity of light and n_L

the apparent index of refraction which will be discussed later.

If the lens has a length d, the time required for the phase to traverse the lens will be $t_L = \frac{d}{v_L}$, while the time required for the external field to sweep over the outside of the lens will be:

$$t = \frac{d}{c}$$

Previous discussion has shown that the difference between these two times should be an odd number of half periods, the more practical design generally being obtained if this difference is equal to one-half period. So if γ is the frequency of the received energy, the following can be written;

$$t_L - t = \frac{1}{2}\gamma \tag{21}$$

and solving for d the elementary lenses are:

$$d\ max = \frac{C}{2\gamma(n_L-1)} = \frac{\lambda}{2(n_L-1)} \quad or \quad \frac{d\ max}{\lambda} = \frac{1}{2(n_L-1)} \tag{22}$$

measured in wave lengths, is the position of the first peak in the gain. Where both ends of the lens are flat, the actual position of the peak may be displaced slightly by the reflection due to Snell's law. The rear end of the lens may be used as an impedance matching device or mode transformer. Then the position of this part, with reference to the metallic guide, will also have a bearing on the position of the maximum peak. Reference is made, for instance, to Fig. 4. The calculated position of the first peak is 1.1 λ which is verified.

The amplitude of the gain is a much more complex problem. It depends on the directivity or pattern of the lens, as well as on that of the receiver to which the lens is fitted. Thinking only of the maximum gain from a given cross section, a first approximation for the relative gain of a lens over that of a metallic mouth of the same cross section may be obtained by considering the lens made up of a number of uniform oscillators arranged as an end-fire array. Each consecutive section of the lens is fed as if the lens constituted a distribution line. Under these conditions the gain derived by classical methods is:

$$G_R = 1 + \frac{d\ max}{\lambda} = 1 + \frac{1}{2(n_L-1)} \tag{23}$$

where G_R is the gain of a lens of uniform cross section over that of a metallic mouth of the same cross section and d max is the lens length affording maximum gain for such a cross section! Experimental verification is fair, as shown on Graph No. 5.

Although this formula is only the roughest kind of an approximation, it is rather interesting in that it shows the gain of a lens to increase, as $n_L \longrightarrow 1$. This can also be deduced from a study of lens patterns and will be discussed again later.

As a very rough approximation it could be said that together a unit area of the two walls normal to the E vector gather the same amount of energy as the lens face normal to the wave propagation. In other words, if the optimum length of a lens is three times its aperture, it will show a gain of four when compared to the bare mouth. This rule of thumb applies only to low loss dielectrics and is on the conservative side. The difficulty of formulating the gain in a rigorous manner is brought out in the following:

The data discussed so far leads to speculation on the limit of relative gain of an ideal lossless lens that by some artifice could be designed with the phase velocity of light. Primarily, it could be said that there is no limit to the gain of such a lens. Experience has shown that a lens will gather more energy than that contained in a square wave length. The energy is not limited by any quantization. However, it is believed that the field depletion caused by the lens would be a limiting factor and that there is a finite limit to either the absolute or relative gain of such an ideal lens as has been postulated.

Using Schelkunoff's method of developing the radiation pattern of a group of sources, which has been so ably performed for lenses by Dr. Horton[3] this formula is:

$$M_0{}^2 \ G_R = M_0{}^2 + /R(d)/^2 \tag{24}$$

in which:

$$R(d) = \int_0^d M(\text{SIN } K^1{}_z + \delta)e^{LKz} \, dz \tag{25}$$

where M_0 and M represent magnetic currents at the origin and along the lens considered as a radiator:

$$K' = \frac{2\pi}{\lambda_L} \qquad K = \frac{2\pi}{\lambda} \tag{26}$$

and δ is a phase angle which can be considered zero, if $K' \longrightarrow K$

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[3]Horton: "On Dielectric Rod As An Antenna", CM-272, UT/DRL No. 66.

There is no apparent constant value for λ_L along a lens, but it decreases slightly as a lens is made long. Admitting, for the present purpose, that this variation is due in reality to field depletion, and that a lens has a characteristic wave length, λ_L, which has been discussed previously, the field depletion can be assumed to follow such a law as:

$$M^2 = M_0^2 \, e^{-2mz} \qquad (27)$$

Putting this into R(d) and integrating:

$$R(d) = \frac{M_0}{a^2+K'}2 \left[e^{\,ad}(a \sin K'd - K' \cos K'd) + K' \right] \quad (28)$$

where "a" is complex. Drop M_0^2, as it appears in terms of G_R, and calculate $R(d)^2$ REAL as:

$$R(d)^2 \text{ REAL} = (\frac{1}{a_1^2 + a_2^2})^2 (a_1^2 - a_2^2)(K'^2 {}_{-e}^{\,-2\,md}(b_2^2-b_1^2) - 2K' \, b_1 \, e^{-md}) \qquad (29)$$

with:

$$a_1 = K'^2 - K^2 + m^2, \qquad a_2 = 2 \, Km, \qquad (30)$$

and b_1 and b_2 are trigonometric functions of Kd and K'd, without significant influence in front of e^{-md} except to note that $b_2^2 - b_1^2$ is negative in the region of small d. Hence, the term in e^{-2md} will be found to bolster up the relative gain for small $\frac{d}{\lambda}$ which is observed experimentally in Fig. 5. If it can be assumed by anticipation that:

$$0 < m \ll K \qquad (31)$$

and passing to the limit of d and allowing $K' \longrightarrow K$ the result would be:

$$G_R \text{ lim } {}^{-1} \sim \Big/ \frac{1}{4 \, m^2} \Big/ \qquad (32)$$

which is finite although quit large.

To get an idea of m, a very thin lens as in Fig. 6, may be used. The optimum length of this lens is 250 λ so that in the lengths considered of the order of 10 λ, phase variations can be neglected. If this lens were excited uniformly throughout its length, each wave length of it would provide an increment relative gain of unity. Its relative gain would lie on the line:

$$G_R - 1 = \frac{d}{\lambda} \qquad (33)$$

If dielectric losses are neglected, it can be admitted that actually each increment of energy gathered by a wave length section is a fixed fraction, α, of the preceding one.

The energy of the first section would be E_1, that of the second α E_1, that of the third $\alpha^2 E_1$, and so on. The relative energy gain of the whole lens of $\frac{d}{\lambda}$ sections would be:

$$\left(1 + \alpha + \alpha^2 + \cdots \cdots \alpha^{\frac{d}{\lambda}-1}\right) = \frac{1 - \alpha^{\frac{d}{\lambda}}}{1 - \alpha} = 7.5 \quad (34)$$

from experiment. The expression $1 - \alpha \sim \frac{1}{7.5}$ is connected to the coefficient of absorption m by $e^{-2m} = \alpha$ and it is concluded that the limit of relative gain is of the order of 45.

This particular lens cross section is about a fifth of a square wave length. An ideal lens of this cross section would gather the energy contained in 9 or 10 square wave lengths or have an ideal aperture of about 3λ. The smallest half power beam width of a lens of less than a wave length would therefore be one third of a radian, or about 19°, which checks the sharpest patterns obtained at this time (Fig. 16).

In the above deductions, dielectric losses were disregarded. Had these been included, the optimum beam width would have turned out slightly sharper than the best experimental patterns have shown.

The ideal lens aperture will be considered again under patterns of arrays. The above discussion shows that the use of long lenses packed closer than several wave lengths apart is not to be recommended.

APPARENT INDEX OF REFRACTION

Before attempting any theorizing on the apparent index of refraction a description of a novel phase meter used to determine this quantity experimentally is given. This meter is shown in the attached Fig. 7. Its principle is to set up a standing wave in a slotted section of wave guide between a reference signal and a signal that has traversed the sample. Both of these signals originate from the same approximately plane wave generated by an emitter at some distance from the receiver. The slotted section is fed from each end by a small transfer probe that traverses resistive wadding so that multiple reflections in the wave guide are attenuated. The relative effect of the sample is measured by noting a node position with the sample in place; with reference to the position of the same node without the sample. Node positions can be accurately gauged by averaging the position of two equal amplitudes indicated by the sliding probe on each side of the desired node.

The apparent index of refraction of the sample can be readily calculated as follows: First, measure a node without the lens. Let l_1 be the distance of this node from the mouth of the reference horn and l_2 its distance from the sample horn. Evidently:

$$l_1' + l_2' = L \qquad\qquad (35)$$

(total electrical length)

Consider the origin of the plane wave at a distance d in front of each horn. Let c be the velocity of light and v the velocity of the signal throughout the wave guide system (assumed uniform in the slotted section). The time for the signal to reach the position defined by l_1' will be $t_1' =$

$\frac{d}{l} + \frac{l_1'}{v}$ and that for the position defined by l_2' will be $t_2' = \frac{d}{c} + \frac{l_2'}{v}$

If our node is close to the electrical center, these two times will differ by one-half period or:

$$\frac{l_1'}{v} = \frac{l_2'}{v} + \frac{\lambda}{2c} \qquad\qquad (36)$$

or

$$\frac{1}{v}\left(l_1' - l_2'\right) = \frac{\lambda}{2c} \qquad\qquad (37)$$

using

$$\frac{1}{v}\left(l_1' + l_2'\right) = \frac{L}{v} \qquad\qquad (37)$$

and adding this becomes:

$$\frac{2}{v}l_1' = \frac{L}{v} + \frac{\lambda}{2c} \qquad\qquad (38)$$

By placing the lens of length d, in front of one of the wave guides, and letting l_1 and l_2 define the new position of the same node we have:

$$t_1 = \frac{d}{c} + \frac{l_1}{v} \qquad\qquad t_2 = \frac{d}{v_L} + \frac{l_2}{v}$$

v_L = unknown lens velocity.

These two differ by one-half period so that;

$$\frac{d}{c} + \frac{l_1}{v} = \frac{d}{v_L} + \frac{l_2}{v} + \frac{\lambda}{2c}$$

or:

$$\frac{1}{v}\left(l_1 - l_2\right) = d\left(\frac{1}{v_L} - \frac{1}{c}\right) + \frac{\lambda}{2c}$$

using as before:

$$\frac{1}{v} (l_1 + l_2) = \frac{L}{v}$$

and adding the result becomes:

$$\frac{2}{v} l_1 = d \left(\frac{1}{v_L} - \frac{1}{c}\right) + \frac{\lambda}{2c} + \frac{L}{v} \qquad (39)$$

Now the measurement gives the positive displacement of the node $(l_1 - l_1')$. Forming the difference:

$$\frac{2}{v} (l_1 - l_1') = d \left(\frac{1}{v_L} - \frac{1}{c}\right)$$

is obtained. Introducing the relations $n = \frac{c}{v}$, and by analogy, defining n_L as $n_L = \frac{c}{v_L}$ and $\lambda g = \frac{v}{\nu}$ (known wave length in guide) the apparent index of refraction is:

$$n_L = 1 + \frac{2\lambda}{d} \frac{(l_1 - l_1')}{\lambda g} \qquad (40)$$

In this expression all coefficients except n_L are given by the experimental data.

A comparison of the results of several hundred determinations for lenses of varying dimensions shows that n_L is given with surprising accuracy by the expression:

$$n_L - 1 = (n-1) \, \theta - \left(\frac{\lambda}{\lambda'c}\right)^2 \qquad (41)$$

in which n is the index of refraction of the dielectric and $\lambda'c$ is a characteristic wave length akin to the more usual cut off wave length of a dielectric filled chamber. In the case of a rectangular chamber the latter would be:

$$\lambda'c = \sqrt{\frac{2n}{(\frac{1}{a})^2 + (\frac{m}{b})^2 + (\frac{p}{d})^2}} \qquad (42)$$

with a, b, and d taken as the dimensions in the E, H, and propagation directions respectively, and l, m, and p are any integers.

Figures 2 and 3 of n_L as a function of d show a more or less irregular variation of the apparent index of refraction for small lengths d. This irregularity can be ascribed to the discontinuity of the lens in the vicinity of the metallic mouth. The effect of this variation is noticeable in side lobes of patterns and will be considered under "Patterns".

As d increases, the apparent index of refraction assumes a constant value almost independent of d, and it seems, therefore, that p is nil in the expression of characteristic wave length. Perhaps a more elegant way of stating this would be to say that because of the forced wave in the dielectric of the lens, the number of half-waves "p" in the direction of propagation, is always held to a minimum by the external wave and the ratio $\left(\frac{p}{d}\right)^2$ becomes insignificant as d increases.

A consideration of the number of half-wave lengths "m" in the H direction leads to the conclusion that this must be zero. The dielectric lens is excited by a plane wave with the E vector normal to the b dimension. The only influence that could cause a cancellation of the E force would be conducting side walls, which do not exist. Careful experimental determination of n_L for narrow rectangular lenses seems to indicate that the dimension "b" does have a small influence on the apparent index of refraction. This influence however is much smaller than that of "a".

The dielectric dipoles are excited in the "a" direction by a plane E wave. The number of full half waves in this direction can be only unity, and therefore $\ell = 1$.

The characteristic wave length of a lens reduces, therefore, to:

$$\lambda'_c = 2 n a \qquad (43)$$

for the more usual rectangular lenses where d is a wave length or longer. In the case of cylindrical lenses, the voltage applied around the periphery of the lens must set up a configuration corresponding to the TE_{11} mode for which

$$\lambda'_c = \frac{n \pi a}{1.84} \qquad (44)$$

in which "a" is the diameter of the rod.

Table No. 1 gives the experimental verification of the equation:

$$n_L = 1 + (n-1) \, e^{-\left(\frac{\lambda}{2na}\right)^2} \qquad (45)$$

for several different lenses of widely varying proportions (see also Fig. 8).

The excellent results given by this formula under widely differing conditions lead one to believe it has a firmer foundation than the empirical one derived from these experiments.

The variations of lens wave length discussed above have been ob-

served by Southworth[4], Mallach[5], and others, as well as ourselves. So far, it has not yet been treated theoretically. This is rather unfortunate because an accurate knowledge of this coefficient is essential to formulate basically correct pattern and gain data. The complexity of the problem can be gauged from Schelkunoff's discussion of waves in dielectric rods which has already been mentioned.

DIELECTRIC DEPOLARIZERS

The observation that the phase velocity through a lens depends on the dimension "a" in the "E" plane leads to a useful depolarizing device. It is often desirable to pick up a signal from a polarized wave although the receiver may be rotating or rolling. Consider a rectangular plate of dielectric of a certain thickness and width; a plane wave polarized along its smallest dimension will have very little phase retard, while one polarized along its width will be retarded to a certain extent. Make the length of the dielectric such that the difference between these two retards is one-quarter period. To secure depolarization, the above plate is mounted in a cylindrical wave guide so that its plane forms an angle of 45° with the axis of the crystal probe.

For simplicity consider an incident wave polarized at 90° to the probe. Without the depolarizer the signal would be zero. However, the incident wave will decompose into two components, one strongly retarded, and directed along the width of the dielectric; and the other slightly retarded and directed along the thickness. The amplitude of each will be 0.707 that of the original. The probe will be sensitive to 0.707 components of each of these individual waves. These probe components are equal and opposed in space but orthogonal in time so that they recompose at the probe to give a resultant equal to one-half of the original intensity.

In practice the depolarizing plate is placed inside the wave guide so as to avoid disturbing the lens pattern that is generally determined by other considerations.

In reality the wave seen by the crystal is circularly polarized. An observer looking down the wave toward the crystal, and seeing the plane of the dielectric rotated clockwise 45° from the crystal axis would see the electric vector rotating counter-clockwise. If the plane of the dielectric were seen at an angle of 45° from the crystal axis is a counter-clockwise direction, the observer would see the electric

[4] Southworth, "Some Fundamental Experiments with Waveguides". Proc. Ire. July, 1937. Volume 25.

[5] P. Mallach. "Dielectric Radiators for DM and CM Waves". Horton, Loc. cit.

vector rotating clockwise.

If a receiver is equipped with one of these devices, and another identical unit is placed on a sender, twice the energy is obtained compared to the use of a sending horn of equivalent beam width. If one of the devices is of opposite hand from the other, no signal is received.

The depolarizer may form a flattened extension of the lens which tapers down in a fish tail shape as shown in Fig. 19. External depolarizers can be made on the same principle. They tend to give unsymmetrical patterns.

It has been possible to check this simplified theory on the phase meter and the attached Fig. 9 has been obtained. The probe signal lags by $\frac{\pi}{4}$ radians when the E vector is in the plane of an operating fish tail, and leads by $\frac{\pi}{4}$ radians when the E vector is perpendicular to the plane.

This experiment gives a striking confirmation of the absence of a cut-off point in a dielectric. If a cut-off existed as for dielectric filled metallic guides, the operation of the depolarizer could still be explained by one component remaining for a short distance only in the dielectric, but the phase lag would be opposite from that observed.

<div align="center">PATTERNS</div>

The pattern of lenses cannot be analyzed exactly because of the lack of a definite formulation of the phase velocity and amplitude along their length. From experimental studies of the apparent index of refraction, it can be assumed that the wave length varies somewhat throughout the length of the lens as well as the intensity of the displacement currents. The experimental data on the index of refraction is a weighted average of the lens wave length. Pattern formulae based upon constant displacement currents and uniform weighted wave lengths will, therefore, not be far off in the main lobe, while data on the side lobes derived with this approximation will present more relative errors.

On the other hand, an exact formulation of the wave length and intensity along the lens would lead to complicated expressions for patterns that would be difficult to handle if not impossible of solution. In practice the patterns of arrays of oscillators are always solved on the assumption of uniform spacing and amplitude, and even these simplified expressions are sufficiently complex to discourage most practical workers.

That the exact formulation of lens patterns is so complex is regrettable because this would lead directly to a general expression of absolute gain.

In spite of the reservations just mentioned, the usual array patterns can be modified, as would be expected from lens behavior, and useful lens pattern formulae derived.

E PATTERN OF A RECTANGULAR LENS

The general formula for arrays established by Stratton is:

$$E^2 = (A \ F_0 \ F_1 \ F_2)^2 \qquad (46)$$

in which F_0 is a form factor depending on the elementary current distribution. As already stated the displacement current in a lens must be a sine wave, zero at the two boundaries and maximum in the middle. The form factor for such a current element 'a the E plane is:

$$F_0{}^2 = \left(\frac{\cos (\frac{\pi}{2} \cos \theta)}{\sin \theta} \right)^2 \qquad (47)$$

where θ is the angle between the E vector and the direction considered. This factor can be checked experimentally by taking the pattern of a dielectric filled wave guide after covering the metal edges with absorbing material so as to reduce as much as possible the diffraction from the metal edges. This device is only partially effective so that some errors from diffraction can be expected. However, experiment verifies the general form of this factor as shown on Fig. 10(A).

If instead of a sine distribution, the only other logical alternate, a constant field, had been used the well known Huygens pattern would have been obtained:

$$(1 + \sin \theta)^2 \qquad (48)$$

which is not verified by experiment.

The next factor F_1 is due in the case of lenses to the side wall absorption as the lens length increases up to a wave length. The fact that it is believed that the side walls are effective energy absorbers suggests that this factor should be that of an orifice in an absorbing screen, namely;

$$F_1{}^2 = \left(\frac{\sin \frac{a\pi}{\lambda_L} \cos \theta}{\frac{a\pi}{\lambda_L} \cos \theta} \right)^2 \qquad (49)$$

Verification of patterns of wave length lenses represented by:

$$E^2 = (F_0 \ F_1)^2 \qquad (50)$$

is given in Fig. 10(E).

At this point one of the limitations of this method appears. Experimental lens patterns progress smoothly from one type of pattern to the other, while formulae derived from metal arrays will only apply to lenses of integer wave length. However the results obtained do have practical use because the transition from one pattern type to another is gradual. It is only necessary to assign the nearest integer number of wave lengths to a lens to get satisfactory data.

The factor F_2 for multiple wave length lenses appears to exist as unity in the expression of the single wave length lens. As the length d increases F_2 departs from unity in a smooth transition and side lobes begin to appear, while the main lobe becomes narrower. The lens then resembles an end-fire array, the factor F_2 of which can be derived from the classical value of:

$$F_2 = \frac{\sin p \frac{\gamma}{2}}{\sin \frac{\gamma}{2}} \qquad (51)$$

where:

p = number of oscillators in array

$$\gamma = \frac{2\pi l}{\lambda} \cos \psi - \alpha \qquad (52)$$

with l = spacing between oscillators, one wave length

$$\psi = \frac{\pi}{2} - \theta$$

α = phase lag from one oscillator to next = 2π

The length l appears actually to be a function of θ. The aperture presented by a lens normal to a plane field is a cotangent function of the angle θ. The wave length must vary inversely to the aperture in the E plane according to some complicated law. A satisfactory expression is obtained if it is admitted that

$$l \sim \frac{1}{a} \sim \lambda_L \sin \theta \qquad (53)$$

where λ_L is obtained by the phase meter or calculation discussed previously. Expression No. (52) becomes:

$$\gamma = \frac{2\pi}{n_L} \cos \psi \sin \theta - 2\pi = \frac{\pi}{n_L} \sin^2 \theta - 2\pi$$

and factor F_2 for a lens "p" wave lengths long becomes:

$$F_2 = \frac{\sin(\frac{\pi p}{n_L} \sin^2 \theta)}{\sin(\frac{\pi}{n_L} \sin^2 \theta)} \tag{54}$$

which after introduction in

$$E^2 = (F_0 \, F_1 \, F_2 \,)^2 \tag{55}$$

is verified experimentally in Figs. 10(C) and (D).

If the length "1" separating two oscillators is assumed to be constant and independent of the angle "θ" an expression;

$$F_2 = \frac{\sin (\frac{\pi p}{n_L} \sin \theta)}{\sin (\frac{\pi}{n_L} \sin \theta)} \tag{56}$$

is obtained which does not verify the experimental results nearly as well as expression No. (54).

To obtain the coefficient F_2, it has been assumed that the lens contained an integer number of wave lengths, and that p is integer. Later on this coefficient F_2 with p unlimited will be made use of. Note here that the reasoning fits better the classical antenna theory if p is limited to integers, but actual lenses know no such limitation. Their patterns progress smoothly from that of the metallic structure to the typical narrow beam with minor side lobes, which merge and vary smoothly as the lens length increases. The coefficient F_2 is found to correspond to this behavior also for p non integer but greater than one. In the following, therefore, p will not be limited to integer but will be assumed to be the ratio $p = \frac{d}{\lambda_L} > 1$.

H PATTERNS OF RECTANGULAR LENSES

The influence of the dimension of a rectangular lens in the H plane is not nearly as pronounced as that of the dimension in the E plane. Experimental determinations of the variation of the phase velocity in function of the H dimension indicate that the field across a lens is very nearly, if not quite, constant as discussed in the determination of the apparent index of refraction "n_L".

Therefore, following the same approach used for the E pattern, the coefficient F_0 will be:

$$F_0 = (1 + \sin \phi) \tag{57}$$

where ϕ is the angle between the magnetic vector and the direction con-
sidered. The H pattern of a rectangular wave length lens then becomes:

$$E_H^2 = (1 + \sin \phi)^2 \left(\frac{\sin \frac{\pi b}{\lambda L} \cos \phi}{\frac{\pi b}{\lambda L} \cos \phi} \right)^2 \qquad (58)$$

which holds nicely as long as "b" is greater than a wave length as shown
in Fig. 11. This formula also represents, with fair accuracy, the pat-
tern of a wide horn if λ_L is replaced by the free space wave length λ,
as also shown in Fig. 11.

For narrow lenses, experimental results do not check the above ex-
pression with the required accuracy. For these lenses, if the field a-
cross the width of the lens is assumed to vary as is the case for the E
pattern, the resulting formula is:

$$E_H^2 = \left(\frac{\cos (\frac{\pi}{2} \cos \phi)}{\sin \phi} \right)^2 \left(\frac{\sin \frac{\pi b}{\lambda L} \cos \phi}{\frac{\pi b}{\lambda L} \cos \phi} \right)^2 \qquad (59)$$

experimental verification of which is shown in Fig. 12, both for very
short and for wave length lenses.

The logical choice of F_2 is then:

$$F_2 = \frac{\sin(\frac{\pi p}{n_L} \sin^2 \phi)}{\sin(\frac{\pi}{n_L} \sin^2 \phi)} \qquad (60)$$

which works equally well for narrow and for wide lenses where $p = \frac{d}{\lambda L} > 1$.

This factor is in contradiction with the known relative independence
of the lens phase velocity and "b". If, as implied by the expression of
E_H in narrow lenses, the phase velocity in a lens should be assumed to
have the same dependence on "b" as on "a" the depolarizer could not oper-
ate. However, experimentally, the variation of velocity with "b" is
small. The framework of the formula for characteristic wave length,
that enters into the expression of n_L, is incapable of interpreting a
small variation of field across the face "b", which apparently exists to
a significant extent in narrow lenses, to form the patterns obtained ex-
perimentally.

Better data on the region of $b > \lambda$ where this effect takes place
could be obtained from patterns of wide long lenses. However, the

latter are so narrow that experimental errors hide the preference that
may exist for:

$$F_2{}^2 = \left(\frac{\sin \frac{\pi p}{n_L} \sin \phi}{\sin \frac{\pi}{n_L} \sin \phi} \right)^2 \tag{61}$$

which would follow from complete independence of n_L and b, or for;

$$F_2{}^2 = \left(\frac{\sin \frac{\pi p}{n_L} \sin^2 \phi}{\sin \frac{\pi}{n_L} \sin^2 \phi} \right)^2 \tag{62}$$

which would be correct if n_L varied strongly with "b".

<div align="center">PATTERNS OF CYLINDRICAL LENSES</div>

The patterns of cylindrical lenses can be determined in the same way
as those of rectangular lenses.

On examining the latter, both the E and H planes are found to be
symmetrical except for the coefficient "F_0" which is chosen to represent
either a sine distribution or a uniform field as the case may be.

In a cylindrical lens it is evident that there must be a sine dis-
tribution in both planes because of the tapering off of the fields at
the edge of the lens, so that the E and H patterns of a cylindrical lens
are identical. This is a consequence of the theory of dielectric wires
already discussed and is borne out entirely except in very short lenses
where the unsymmetric diffraction from the metallic mouth has enough im-
portance to disturb the lens patterns. This diffraction is, however, not
as troublesome in a circular mouth as in a rectangular one.

The factor F_1 will be changed to;

$$F_1 = \frac{J_1 \frac{\pi a}{\lambda L} \cos \theta}{\frac{\pi a}{\lambda_L} \cos \theta} \tag{63}$$

as derived by Schelkunoff[6] for a circular orifice in an absorbing screen.
Otherwise the coefficients for a cylindrical lens are the same as those
of a rectangular lens, namely;

$$E_E{}^2 = E_H{}^2 = \left(\frac{\cos \left(\frac{\pi}{2} \cos \theta \right)}{\sin \theta} \right)^2 \left(\frac{J_1 \frac{\pi a}{\lambda_L} \cos \theta}{\frac{\pi a}{\lambda_L} \cos \theta} \right)^2 \left(\frac{\sin \frac{p\pi}{n_L} \sin^2 \theta}{\sin \frac{\pi}{n_L} \sin^2 \theta} \right)^2 \cdot \tag{64}$$

[6]Schelkunoff, Loc cit, Page 356.

Figure 13 gives the experimental verification of this expression for several structures.

It may be observed here that the first zero of J_1 occurs for 3.8. If $\frac{\pi a}{\lambda_1} \leq$ 3.8, there will be no side lobes in a cylindrical lens of one wave length.

In fact, the same observation could be made for rectangular lenses. In this absence of side lobes is found one of the fundamental advantages of dielectric lenses over metallic structures where dimensions are limited and side lobes are detrimental.

PATTERN OF POINTED LENS

The preceding discussion of lenses of uniform dimensions emphasizes the fact that the necessary tools to accurately formulate the pattern are not available even for uniform cross sections. When considering a lens tapered to a point the difficulties are found to be much greater, for both the amplitude and wave length are made to vary much more sharply.

Qualitative data for the tapered part of a lens is all that can be established.

Experience has always shown that tapering a lens down to an edge in the case of rectangular lenses or to a point in the case of circular ones, decreases the side lobes of long lenses and broadens slightly the main lobe. The curve in Fig. 14, of a four wave length pointed lens is typical. It is compared with the pattern of two cylindrical lenses of three and four wave lengths respectively. The main lobe is seen to broaden and the side lobes become practically non-existent, in a pointed lens.

This suggests that the field in the pointed unit operates similarly to that over a parabolic reflector. If the latter is first illuminated with a uniform flux, maximum sharpness and gains are obtained along with side lobes. Now if the field is tapered down in any manner toward the edges, the gaussian distribution for instance, the side lobes decrease, the main lobe increases in width slightly, and the maximum gain decreases.

Schelkunoff[7] indicates an expression:

$$F = \left(e^{i\xi} - e^{i\xi_1}\right)\left(e^{i\xi} - e^{i\xi_2}\right) ---- \left(e^{i\xi} - e^{i\xi_p}\right) \tag{65}$$

for the pattern of any array of oscillators with any amplitudes at fixed spacings. The angles $\xi_{1,2}$ are those at which the pattern is zero.

[7]Schelkunoff, Loc cit, Page 350.
Wolff, I.R.E., May, 1937.

and on introducing these values a polynome of n terms in $e^{\iota(n-1)\xi}$
results, the coefficients of which give the required amplitudes of the
oscillators. This method is difficult to apply because experimentally
the pattern of a pointed lens is zero over a wide region. The equiv-
alent zero points are numerous and the expressions become quite complex.

Another more rigorous treatment is to express the experimental pat-
tern as a Fourier[8] series, and to compare this series with that repre-
senting an array of oscillators of random spacing and amplitude. The
method has been discussed elsewhere and will not be repeated here. The
treatment shown in Fig. 14, is generally sufficient for practical pur-
poses.

Upon measuring the phase of a pointed lens the quantity $(n_L - 1)$ is
found to be almost exactly one-half of that of a cylindrical lens of
similar length. This suggests that only one-half the oscillators in the
point have much influence. Therefore, p in the previously developed
pattern expressions is taken as just half the number of wave lengths of
the pointed lens. With this adjustment a fair representation of the
main beam of a pointed lens is obtained as shown in Fig. 14.

<center>PATTERNS SUGGESTED BY OTHER WORKERS</center>

Excellent work has been done on lens patterns by Mallach[9]. He de-
rives an expression for cylindrical units of small cross sections:

$$E^2 = \left(\frac{\sin \frac{\pi d}{\lambda}(n_L - \sin \theta)}{\frac{\pi d}{\lambda}(n_L - \sin \theta)} \right)^2 \qquad (66)$$

with d = lens length, which is commendable for its simplicity and gives
good results for long thin cylindrical lenses, which were the only ones
considered by him. This expression gives a better approximation of the
side lobes of thin lenses than ours but as Mr. Mallach points out, the
beam widths it gives are broader than those obtained experimentally.
This expression falls down entirely for short lenses, particularly if
they are thick, which is generally the case in our work. In passing we
note that he failed, as well as ourselves, in formulating the apparent
index of refraction n_L. The results of Mr. Mallach are the only recent
published data we know of on wave length lenses.

Where a closer analysis of side lobes is desirable, the use of
Fourier series as developed by Wolff[10] for patterns leads to much better

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[8] Wolff, I.R.E., May, 1937.
[9] P. Mallach: "Dielectric Radiators For DM and CM Waves", Page 14.
            Translated by P. L. Harbury, Harvard University,
            Cambridge, Massachusetts.
[10] Wolff, I.R.E., May, 1937.

approximation than any of those discussed herein. A consideration of all
the experimental patterns shows that these tend to be uniformly zero
over much broader regions than can be reconciled with metallic array
theory. The framework of the expression obtained by this theory does
not permit the interpretation of functions remaining zero over large
regions, while the Fourier series is well adapted to just this type of
function. However, for the central portion of lens patterns, the metal-
lic array theory gives sufficient accuracy for practical purposes at the
expense of much less labor.

LENS ARRAYS

If F^2 represents the pattern of an individual lens, then the pattern
of an array of lenses equally spaced in either the E or H plane and of
equal amplitude will be:

$$E_E{}^2 = F2F_3{}^2 \tag{67}$$

in which:

$$F_3 = \frac{\sin n_3 \frac{\gamma_3}{2}}{\sin \frac{\gamma_3}{2}} \tag{68}$$

with:

$$\gamma_3 = \frac{2\pi}{\lambda} a_3 \cos \theta + \alpha_3 \tag{69}$$

where a_3 is the spacing between the lenses and α_3 the difference in
phase from one unit to its next door neighbor, and θ is the angle meas-
ured from the E or H vector as the case may be.

In the case of only two antennae, with $\alpha_3 = 0$, the normalized
field intensity is:

$$E^2 = F^2 \left(\frac{\sin 2\pi a_3 \cos \theta}{\sin \frac{\pi a_3}{\lambda} \cos \theta}\right)^2 = F^2 \left(2\cos\left(\frac{\pi a_3}{\lambda}\cos \theta\right)\right)^2 \sim \left[\frac{F^2}{2} + \frac{F^2}{2}\cos\left(2\frac{\pi a_3}{\lambda}\cos\theta\right)\right]$$

The half power point of this array (assuming F = constant for small
angles) will be such that:

$$\cos\left(\frac{2\pi a_3}{\lambda} \cos \theta\right) = 0, \tag{71}$$

or

$$\frac{2\pi a_3}{\lambda} \cos \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{K}{2}\pi \qquad \text{K being odd integer.}$$

$$\cos \theta_{1/2} = \frac{\lambda K}{4a_3}$$

Suppose we ask for $\theta_{1/2}$ = 89.75° or a half degree beam width. The corresponding spacing between elements is 57 wave lengths. Such an array would have a total power beam width of only one-half degree. A parabolic reflector of the same aperture with gaussian distribution would have a half power point one degree wide. The two lenses would appear to have a considerable advantage over the metallic reflector, except that the side lobes of the lens system would be very numerous and in the vicinity of the main lobe they would be practically as intense as the latter. The lens array, therefore, would be useless from a practical standpoint with this arrangement.

The envelope of the side lobes of the lens array is given by F^2. This depends principally on the factor:

$$F_2{}^2 = \left(\frac{\sin \left(\frac{p\pi}{n_L} \sin^2 \theta \right)}{\sin \frac{\pi}{n_L} \sin^2 \theta} \right)^2 \qquad (72)$$

and even with the longest practical lens of say ten wave lengths, the first side lobe would be 93% of the principal one.

A four lens array for the same beam width would have an aperture of 79.2 λ. The usual parabolic reflector of the same aperture would have a half power point of 0.725 degrees.

The advantage of the array has been decreased from this standpoint by using four elements. The highest side lobe, using a ten wave length lens will be about 6%, whereas the parabolic dish would have no side lobes.

This analysis has been continued and shows that the dielectric lens array for very narrow beams and low side lobes has no practical advantage over the metallic reflector or the equivalent metallic lenses. The lens array becomes as bulky as the metallic structures.

On the other hand, in a lens array designed for moderate beam widths the spacing becomes so small that the individual elements are apt to interfere with each other by depleting the field in their immediate vicinity.

In these arrays, if side lobes are not detrimental, the width of the main lobe can be made about half that of a horn of the same aperture, but the first side lobes will be of the order of 20%. As an example, a two element array using four wave length tapered units spaced 3.43 λ apart will have a main lobe 8° wide as compared to about 17° for the equivalent horn, but the first side lobe occurs 17° off center and is 25%. To reduce this side lobe would require longer elements which defeat their purpose by interference.

The lens array or polyrod array has, therefore, practically no advantage over metallic structures.

For moderate beam widths single lenses do the job better than an array and are much simpler. For extreme beam sharpness and few elements, the side lobes of the lens array are as important as the main lobe. To reduce the side lobes requires the use of many elements and the structure eventually becomes as bulky as the equivalent metallic structure.

The attached graphs, Figs. 16, 17, 18, and 20, give several practical single lenses achieving moderate beam widths with low side lobes. These are considered the limit of practical application of dielectric lenses.

GAIN FROM PATTERNS

Before leaving the question of lens patterns, an attempt can be made to see how the absolute gain of a lens should vary with its dimensions. Take the complete pattern expression:

$$E^2 = A^2 \, F_0{}^2 F_1{}^2 \, F_2{}^2 \tag{73}$$

for a rectangular lens. Consider only the gain on axis where $\theta = \dfrac{\pi}{2}$. The factors become successively:

$$F_0{}^2 = \left(\frac{\cos(\frac{\pi}{2} \cos \theta)}{\sin \theta} \right)^2 = 1 \tag{74}$$

$$F_1{}^2 = \left(\frac{\sin \frac{\pi a}{\lambda_L} \cos \theta}{\cos \theta} \right)^2 = \frac{0}{0} = 1 \tag{75}$$

$$F_2{}^2 = \left(\frac{\sin \frac{p\pi}{n_L} \sin^2 \theta}{\sin \frac{\pi}{n_L} \sin^2 \theta} \right)^2 = \left(\frac{\sin \frac{p\pi}{n_L}}{\sin \frac{\pi}{n_L}} \right)^2 \tag{76}$$

in which p, the number of wave lengths, is for maximum gain:

$$p \text{ max} = \frac{d \text{ max}}{\lambda L} = \frac{n_L}{2(n_L - 1)} \tag{77}$$

Now n_L can approach unity if the lens is quite small. However, it is evident from the curves of n_L that even for a lossless dielectric the effectiveness of the lens increments decreases as the lens length increases.

Neglecting this consideration for the moment and using the empirical formula for "n_L" , we get:

$$p \text{ max} = \frac{n_L}{2(n_L-1)} = \frac{1+(n-1)e^{-(\frac{\lambda}{2na})^2}}{2(n-1)e^{-(\frac{\lambda}{2na})^2}} \tag{78}$$

Allow "a" to become quite small and the limit of p max becomes:

$$p \text{ max lim} \longrightarrow \tfrac{1}{2}\left[\frac{1}{(n-1)e^{-(\frac{\lambda}{2na})^2}} + 1\right] \sim e^{(\frac{\lambda}{2na})^2} \tag{79}$$

If: $1 < n \leqslant 2$

the value of n_L in function of p maximum is

$$n_L = \frac{2p}{(2p-1)} \tag{80}$$

Putting this in (76) the result is:

$$\frac{\sin \frac{p\pi}{n_L}}{\sin \frac{\pi}{n_L}} = \frac{\sin \frac{\pi(2p-1)}{2}}{\sin \frac{\pi}{2p}(2p-1)} = \frac{\sin (\pi p \frac{\pi}{2})}{\sin (\pi - \frac{\pi}{2p})} = \left|\frac{\cos (\pi p)}{\sin \frac{\pi}{2p}}\right| = \frac{1}{\sin \frac{\pi}{2p}} \cdot \tag{81}$$

if p = integer.

If the lens is such that p maximum = 1, it is necessary that

$$n_L = 2(n_L - 1) = 2 n_L - 2 \quad n_L = 2 \quad \text{or } \mathcal{E} > 4 \tag{82}$$

This case corresponds to rather inefficient lenses. Otherwise, as p becomes large $\sin \frac{\pi}{2p}$ can be replaced by its angle and equation becomes:

$$E^2 = A^2 \frac{(2p)^2}{\pi} \sim \frac{a^4}{\lambda^4} e^{(\frac{\lambda}{2na})^2} \sim \beta(\frac{\lambda}{a}) \tag{83}$$

The maximum gain of a lens is proportional to a function of the inverse of the lens cross section. This is rather a surprising conclusion and quite the contrary that would be expected at first sight. However, it is qualitatively borne out experimentally. Lenses of smaller cross section can be made to pick up larger energies than lenses of large cross sections in the region of $\frac{a}{\lambda} < 1$.

This is a logical consequence of the observation that the side walls or a lens are effective gatherers of energy. The increase of lens energy velocity is rapid as the cross section decreases below a wave length so that the side wall area can be increased by lengthening the lens without unphasing so as to more than offset the reduction due to the decreasing aperture.

This conclusion has to be tempered by the depletion of the external field by long lenses. Logically, this depletion must occur and it can be shown up by several experimental methods. It has been determined that the energy absorbed by a lens can be several times that flowing in a square wave length cross section, but that as the lens length increases the incremental effectiveness of added length decreases. Considering "A" as a constant depending only on the lens cross section is, therefore, not correct.

However, the inverse function of "a" is almost exponential and its effect remains, therefore, preponderant on gain.

This conclusion is confirmed experimentally for small apertures growing up to above a wave length. It does not appear to hold, however, for apertures of several wave lengths, but in this region our whole theory of patterns ceases to be valid. Here the lenses begin to resemble optical units in their behavior.

· By admitting that the relative gain obtained experimentally, namely;

$$G_{rel} = 1 + 2\frac{1}{(n_L - 1)} \tag{84}$$

is approximately correct, this can be converted to absolute gain by multiplying by a coefficient $A = \frac{24}{\lambda^2}$. Roughly the result is a function of the form:

$$G_{abs} = A\frac{a^4}{\lambda^4} + B\left(\frac{\lambda}{a}\right)^2 \quad Z \gg 1 \tag{85}$$

The first term is very large for large $\frac{a}{\lambda}$ and small for small $\frac{a}{\lambda}$ while the second term behaves in the opposite manner.

This is approximately what happens experimentally as shown on the attached Graph No. 15.

The effect of the variation of the dielectric constant of the lens material can be gauged from the preceding expressions.

The gain on axis is proportional to the square of a factor (no. 81)

$$F = \frac{1}{\sin \frac{\pi}{2p}} = \frac{1}{\sin \frac{\pi (n_L - 1)}{n_L}} \qquad (86)$$

With dielectrics for which $\mathcal{E} < 4$, the apparent index is less that two so that;

$$0 < \pi \frac{n_L - 1}{n_L} < \frac{\pi}{2} \qquad (87)$$

and therefore the maximum gain with the usual plastic dielectrics increases as the apparent index of refraction decreases. As this is tied to the true index by

$$n_L - 1 = (n - 1) \; e^{-(\frac{\lambda}{2 n \alpha})^2} \qquad (88)$$

the true and apparent index vary in the same direction and a small dielectric constant is beneficial. This is verified experimentally within the limits of dielectrics used. Higher gains have been achieved with "F 1114" having an index of refraction of 1.4, than with polystyrene $(n = 1.58)$ which in turn is superior to glass $(n \lesssim 2)$. Naturally, if $n \longrightarrow 1$, the lens ceases to exist and entrapment of energy by total reflection does not occur. The equations used do not reflect this effect but consider the angle of total reflection as a constant. The region $n \longrightarrow 1$ has not been investigated by us.

LENSES AS IMPEDANCE MATCHERS

An ideal lens would transform a plane wave in free space to the type of wave existing in the metallic structure to which it is connected. If this were possible, the transition from the metallic structure to space would be achieved smoothly by the lens without reflection. The lens would match the wave guide impedance to that of a plane wave in space.

The use of the standing wave meter and the standing wave ratio to study impedance matching and the use of dielectric slugs to obtain matching is described in the literature and will not be repeated here.

Suffice it to say that a slug of dielectric in a wave guide can be made of sufficient length to match the wave guide impedance to that of atmosphere.

In practice, the lens material is extended into the metallic structure for just such a length, the amount of which can be adjusted, so as to bring the standing wave ratio to unity, as shown on the standing wave meter.

Although theoretically the quality of the lens proper can be judged by its standing wave ratio, in practice this effect is very much less than that of the matching slug.

Therefore, it was best in this work, to calculate the lens length from the formulae for the lens, and to limit the use of the standing wave meter to the adjustment of the matching slug.

The use of an SWR meter to check general lens behavior would be a great simplification over the present laborious methods. Unfortunately, with the exception of the matching device mentioned above, there is little experimental relation between the SWR and the gain of a lens. When the length of a lens has been based on either a calculated or experimental determination of phase velocity, and if the lens is continued into the wave guide in the form of a transformer section, the minimum SWR will indicate the best adjustment of length around that indicated by the phase velocity.

The SWR of a bare tube end is of the order of 2, indicating a reflected voltage of about 33% of that of the primary wave. The energy loss from this reflection is only about 11% and can be easily disregarded in most lens applications.

If the transformer section is placed so as to add its reflection to that of the mouth, the total reflected voltage may be as high as 66%, corresponding to an SWR of five and an energy loss of almost 50%. This is brought out in the attached compared graphs of lenses of small cross section with transformer extensions, Nos. 2 and 3.

Lens No. 2 gave an SWR of less than 1.2 at all peak points of gain and approximately five at nodes of gain. Two symmetric pattern lenses, one 40° and the other 20° wide, will have gains in the ratio of 1 to 4. The SWR meter shows the variation of impedance from one to the other is only slight, and can be hidden by small errors in transformer adjustment. The study of the SWR of lenses does not give very reliable results.

TABLE NO. 1

WAVE LENGTH LENSES

Experimental Verification of

| a	Lens Dimensions in λ b	d	$\dfrac{\lambda^2}{4n^2a^2}$	$e^{-\frac{\lambda^2}{4n^2a^2}}$	Calculated n_L	Experimental n_L	Error Calculated - Experimental
1.206	2.381	.969	.0687	.93360	1.542	1.572	− .03
.826	1.984	3.096	.1467	.86355	1.502	1.477	+ .025
.826	2.578	1.534	.1467	.86355	1.502	1.489	+ .013
.826	4.566	3.096	.1467	.86355	1.502	1.488	+ .014
.397	1.984	1.588	.6349	.53004	1.308	1.315	− .007
.397	2.580	3.969	.6349	.53004	1.308	1.293	− .015
.397	4.566	4.881	.6349	.53004	1.308	1.315	− .007
.318	2.580	3.988	.9920	.37083	1.215	1.208	+ .007
.107	1.984	15.875	8.7040	.00017	1.0001	1.002	− .0019
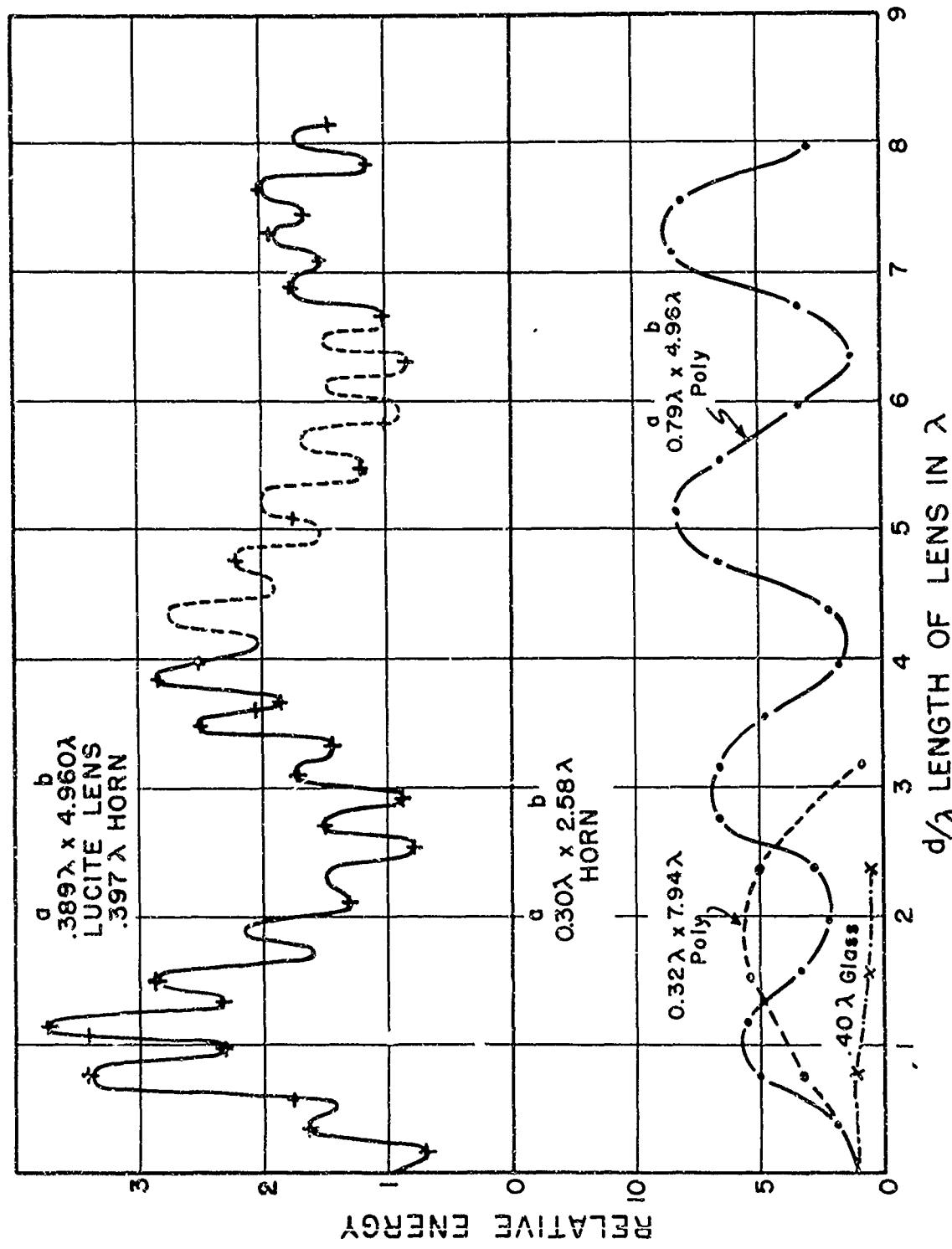
RELATIVE GAINS OF VARIOUS LENS HORN ARRANGEMENTS
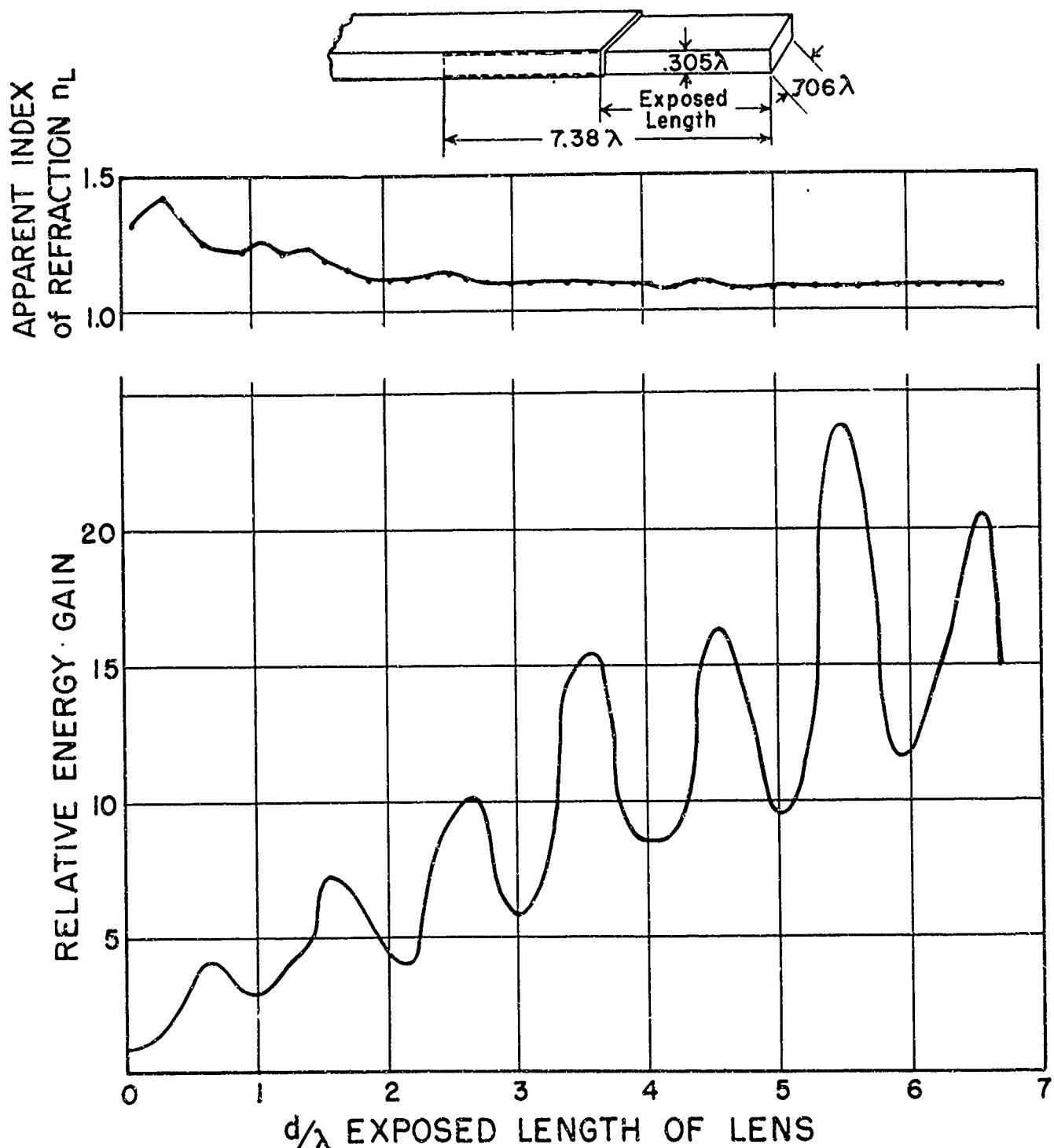(preliminary experiments)

Fig. 1

Fig. 2

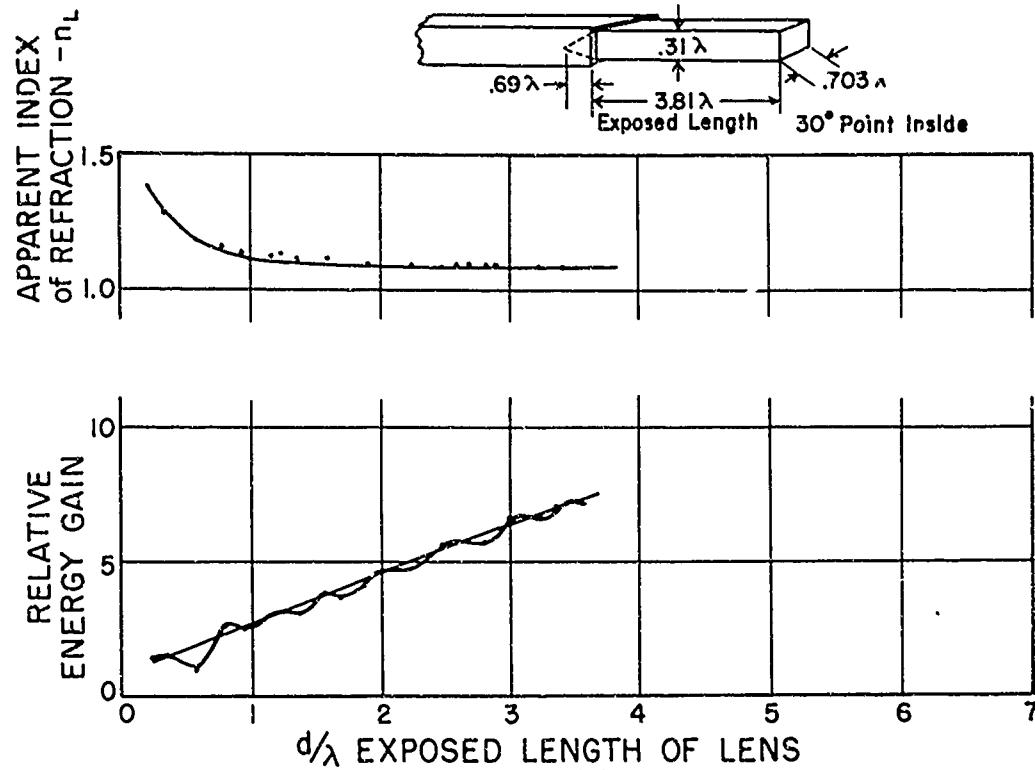LENS WITH TRANSFORMER SECTION
INSIDE WAVE GUIDE

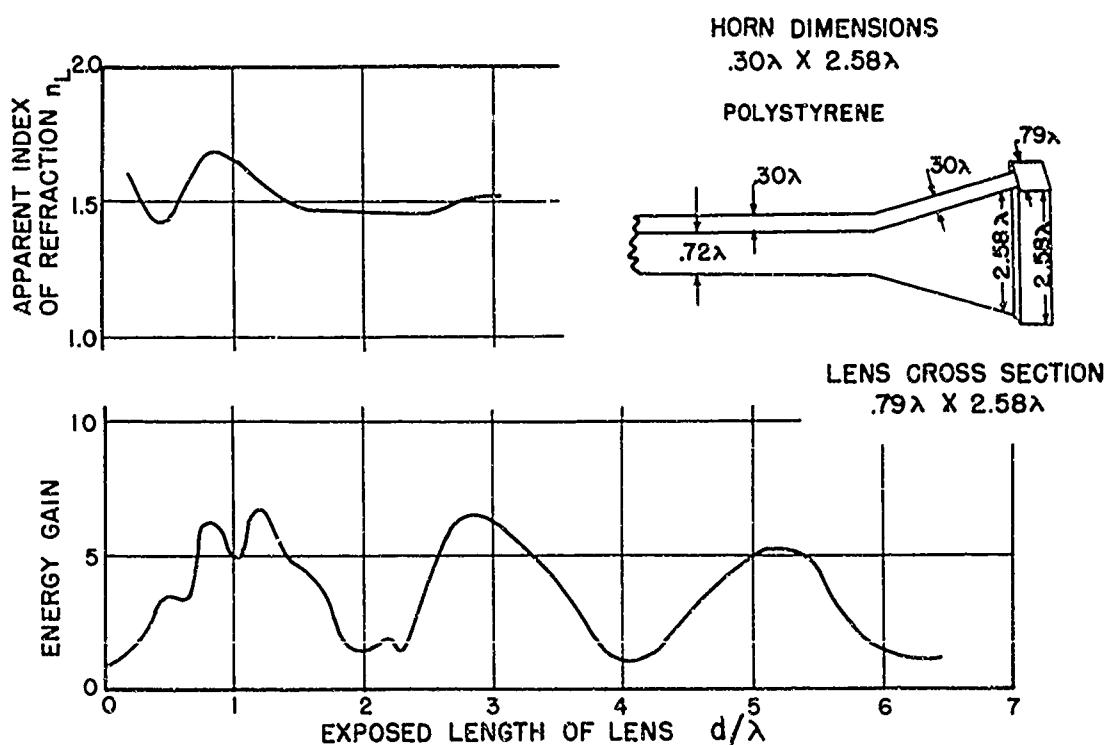Fig. 3

FI114 BLOCKS IN FRONT OF WAVE GUIDE

Fig. 4

LENS AND HORN CHARACTERISTIC

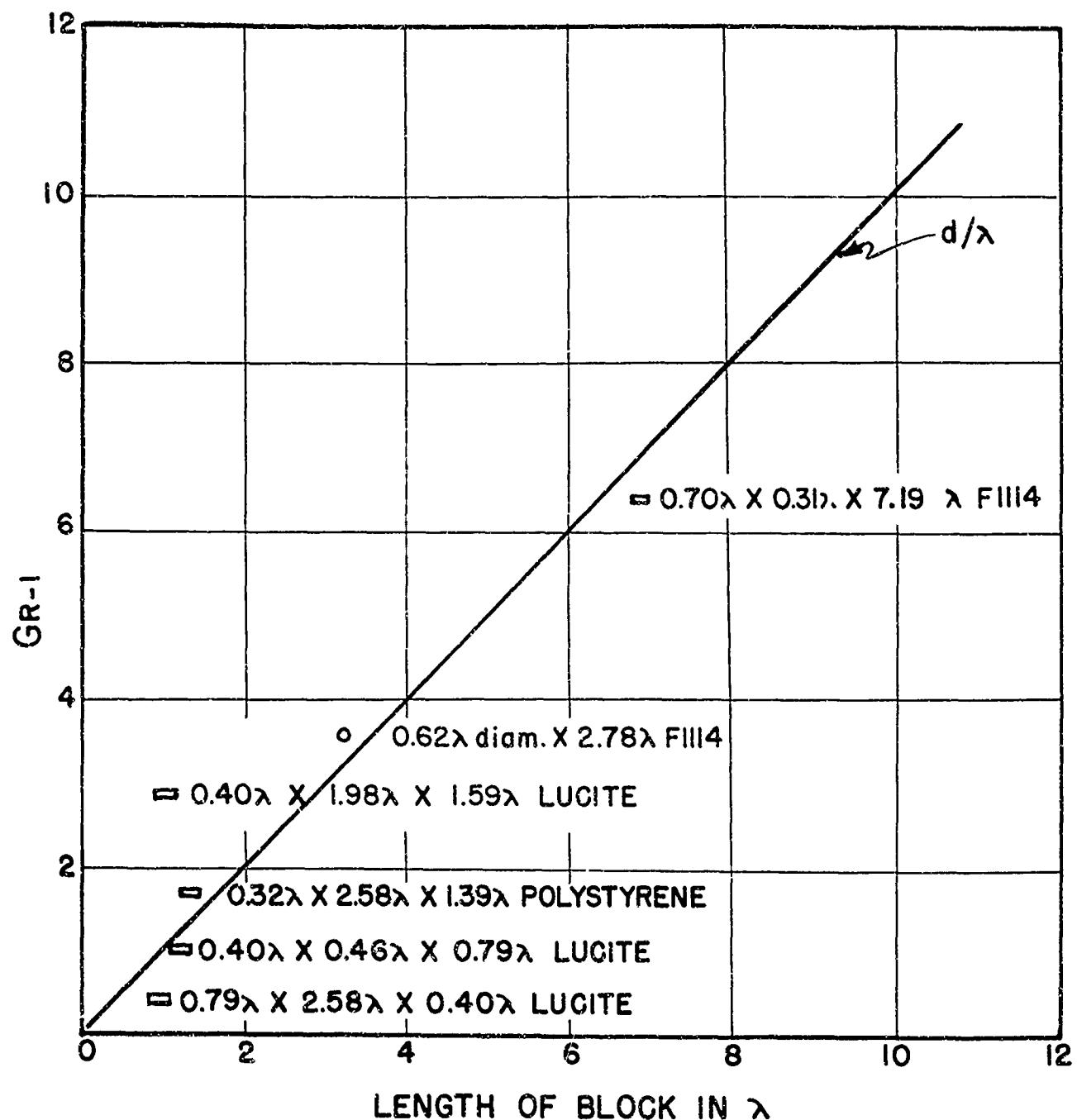## Fig. 5

RELATIVE MAXIMUM GAIN OF A LENS
COMPARED TO METALLIC MOUTH

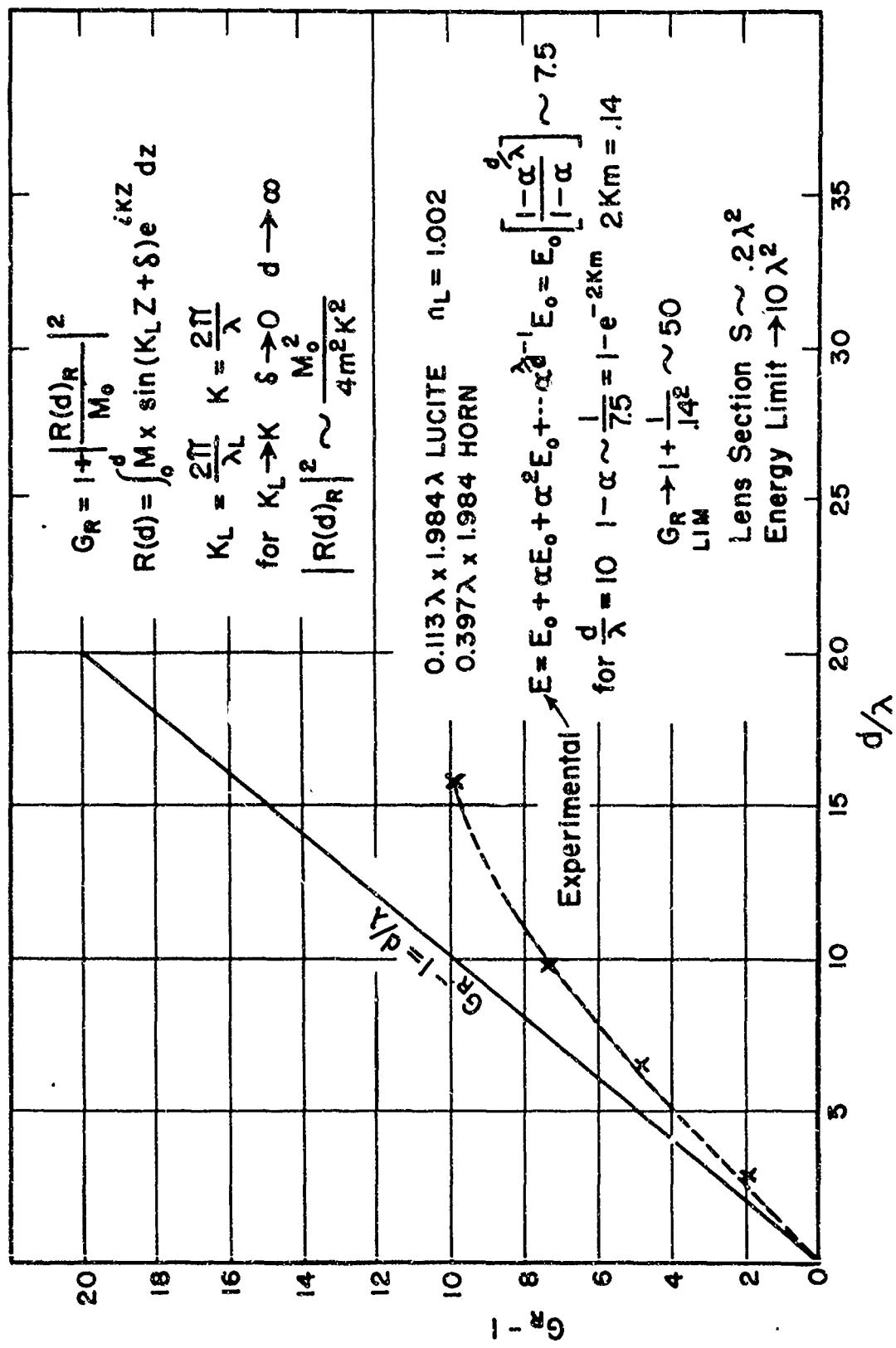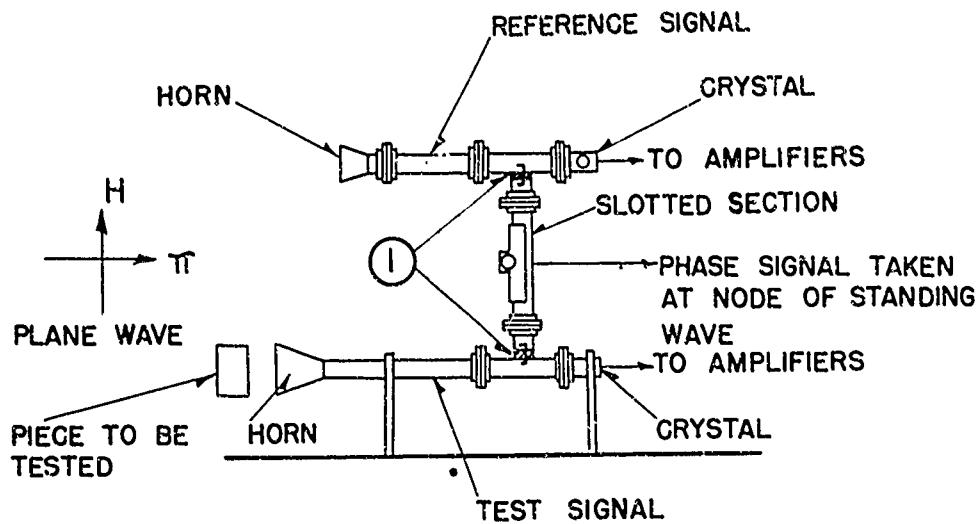$$GR-1 \sim \frac{d_{max}}{\lambda} \sim \frac{1}{2(n_L-1)}$$

$$G_R = 1 + \left|\frac{R(d)_R}{M_o}\right|^2$$

$$R(d) = \int_o^d M \times \sin(K_L Z + \delta)e^{iKZ}\, dz$$

$$K_L = \frac{2\pi}{\lambda_L} \quad K = \frac{2\pi}{\lambda}$$

for $K_L \to K$ $\delta \to 0$ $d \to \infty$

$$|R(d)_R|^2 \sim \frac{M_o^2}{4m^2 K^2}$$

$0.113\lambda \times 1.984\lambda$ LUCITE $n_L = 1.002$
$0.397\lambda \times 1.984\lambda$ HORN

$$E = E_o + \alpha E_o + \alpha^2 E_o + \cdots \alpha^{\frac{d}{\lambda}-1} E_o = E_o\left[\frac{1-\alpha^{\frac{d}{\lambda}}}{1-\alpha}\right] \sim 7.5$$

for $\frac{d}{\lambda} = 10$ $1 - \alpha \sim \frac{1}{7.5} = 1 - e^{-2Km}$ $2Km = .14$

$$G_R \underset{LIM}{\to} 1 + \frac{1}{.14^2} \sim 50$$

Lens Section $S \sim .2\lambda^2$
Energy Limit $\to 10\lambda^2$

Experimental

$G_R - 1 = d/\lambda$

Fig. 6

RELATIVE GAIN OF THIN LENS

REFERENCE SIGNAL

HORN   CRYSTAL

TO AMPLIFIERS

SLOTTED SECTION

H

π

PLANE WAVE

PHASE SIGNAL TAKEN
AT NODE OF STANDING
WAVE

TO AMPLIFIERS

PIECE TO BE
TESTED

HORN

CRYSTAL

TEST SIGNAL

① SMALL COUPLING PROBES BACKED BY
RESISTIVE MATERIAL TO REDUCE
SPURIOUS REFLECTION.

Fig. 7
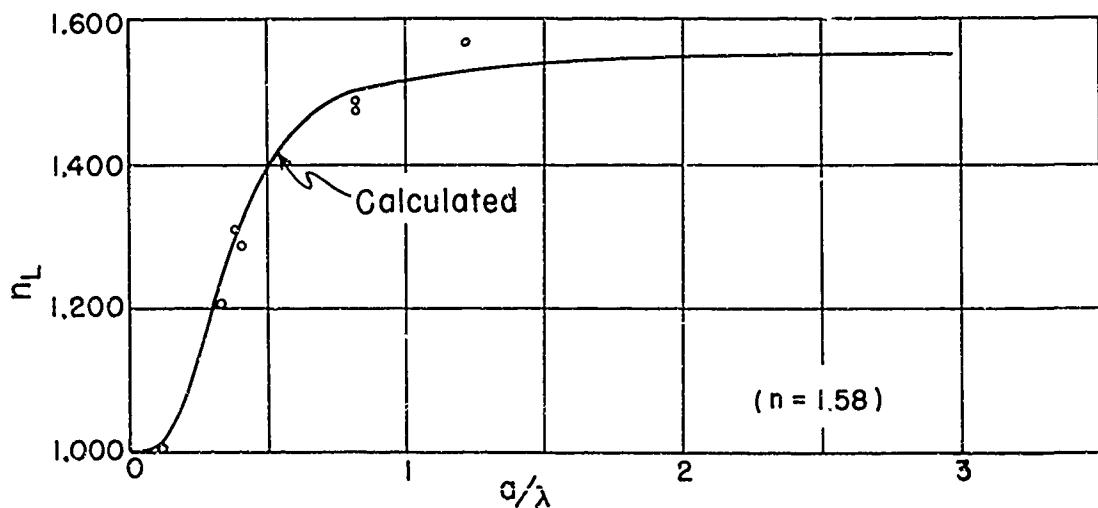## PHASE METER

Calculated

n_L

a/λ

(n = 1.58)

Fig. 8

EXPERIMENTAL VERIFICATION OF $n_L - 1 = (n-1)e^{-(\frac{\lambda}{\lambda'_c})^2}$

For Rectangular Lenses $\lambda'_c = 2na$
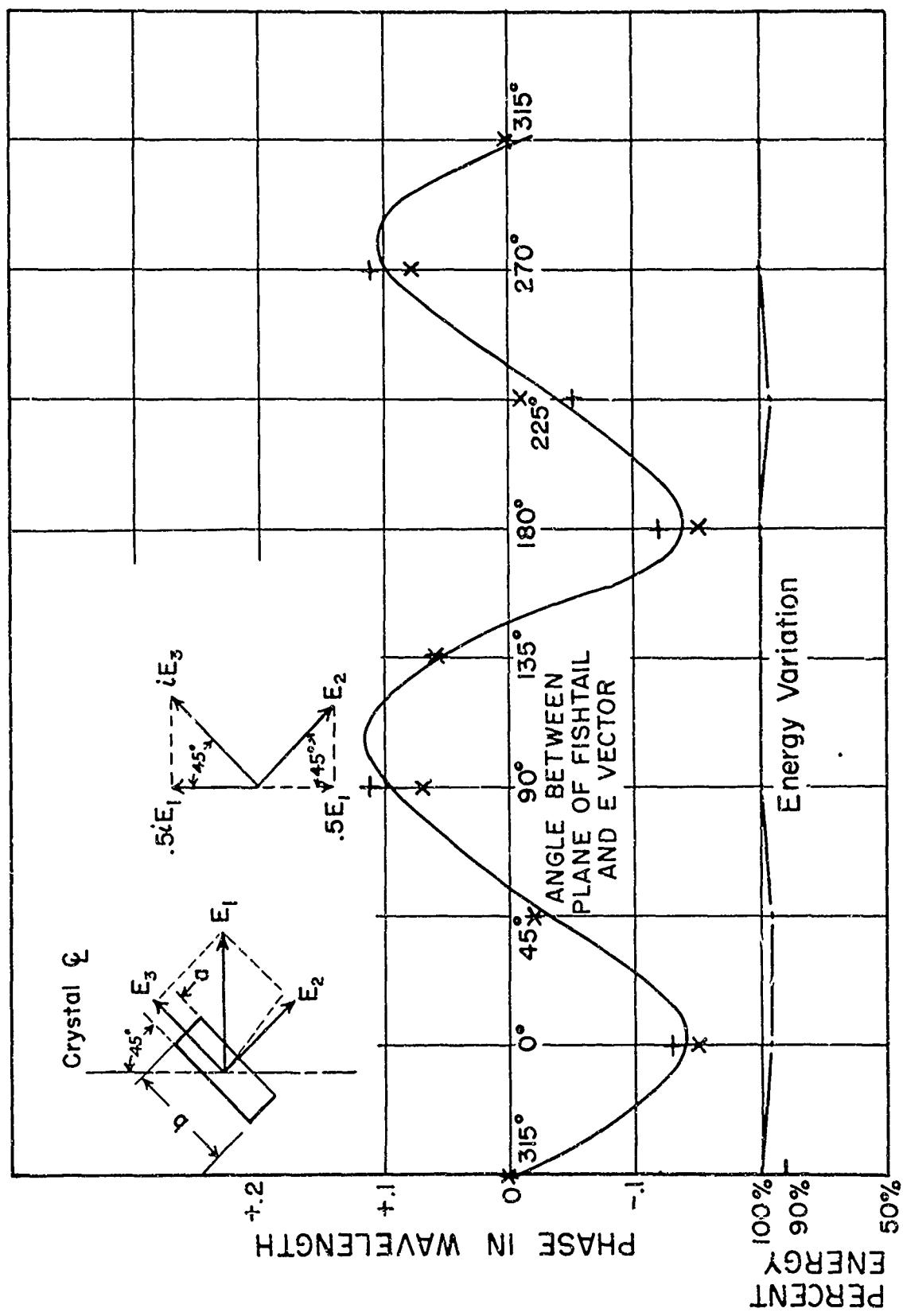
For Cylindrical Lenses $\lambda'_c = \dfrac{n\pi a}{1.84}$

Fig. 9

PHASE VARIATION FROM ROTATION
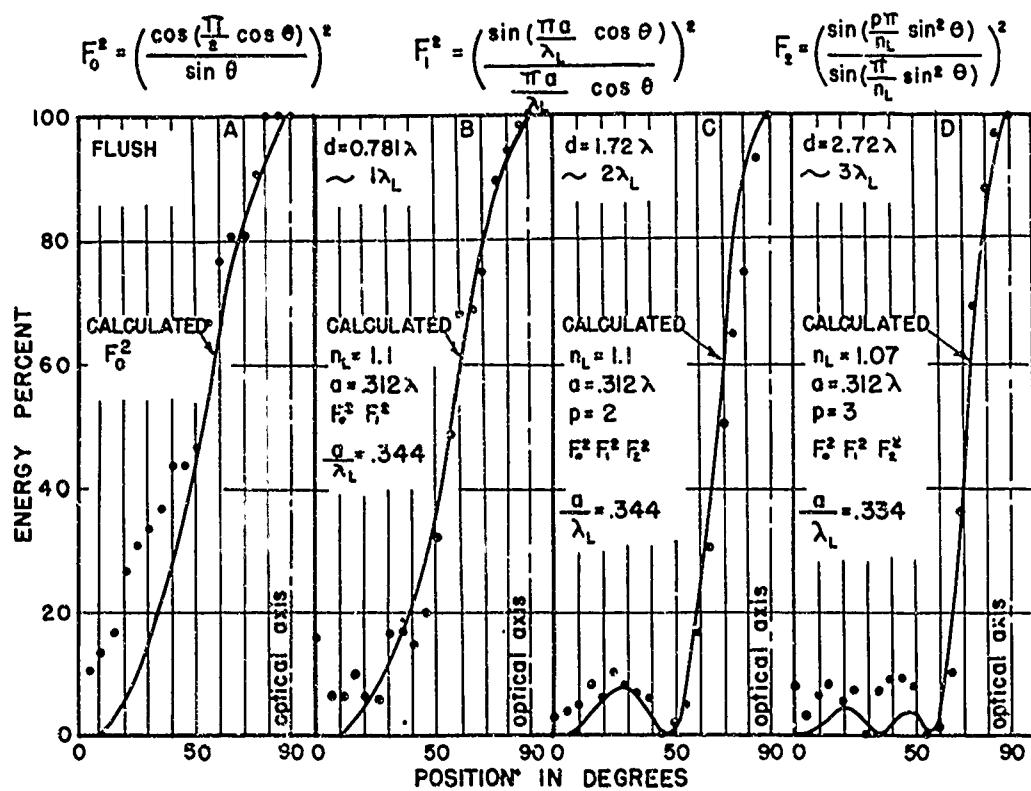OF FISH TAIL DEPOLARIZER

$$F_o^2 = \left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2 \qquad F_1^2 = \left(\frac{\sin\left(\frac{\pi a}{\lambda_L}\cos\theta\right)}{\frac{\pi a}{\lambda_L}\cos\theta}\right)^2 \qquad F_2 = \left(\frac{\sin\left(\frac{p\pi}{n_L}\sin^2\theta\right)}{\sin\left(\frac{\pi}{n_L}\sin^2\theta\right)}\right)^2$$

Fig. 10

NARROW RECTANGULAR FIII4 LENS
E PATTERNS

$$F_o^2\ F_1^2 = (1+\sin\varphi)^2\left(\frac{\sin\frac{b\pi}{\lambda}\cos\varphi}{\frac{b\pi}{\lambda}\cos\varphi}\right)^2 \qquad F_o^2\ F_1^2 = (1+\sin\varphi)^2\left(\frac{\sin\frac{b\pi}{\lambda_L}\cos\varphi}{\frac{b\pi}{\lambda_L}\cos\varphi}\right)^2$$

Fig. 11

WIDE RECTANGULAR LUCITE LENS
H PATTERNS

$$F_0{}^2 = \left(\frac{\cos\left(\frac{\pi}{2}\cos\varphi\right)}{\sin\varphi}\right)^2 \qquad F_1{}^2 = \left(\frac{\sin\left(\frac{\pi b}{\lambda}\cos\varphi\right)}{\frac{\pi b}{\lambda}\cos\varphi}\right)^2 \qquad F_2{}^2 = \left(\frac{\sin\left(\frac{\pi b}{\lambda_L}\cos\varphi\right)}{\frac{\pi b}{\lambda_L}\cos\varphi}\right)^2 \qquad F_2{}^2 = \left(\frac{\sin\left(\frac{p\pi}{n_L}\sin^2\varphi\right)}{\sin\frac{\pi}{n_L}\sin^2\varphi}\right)^2$$
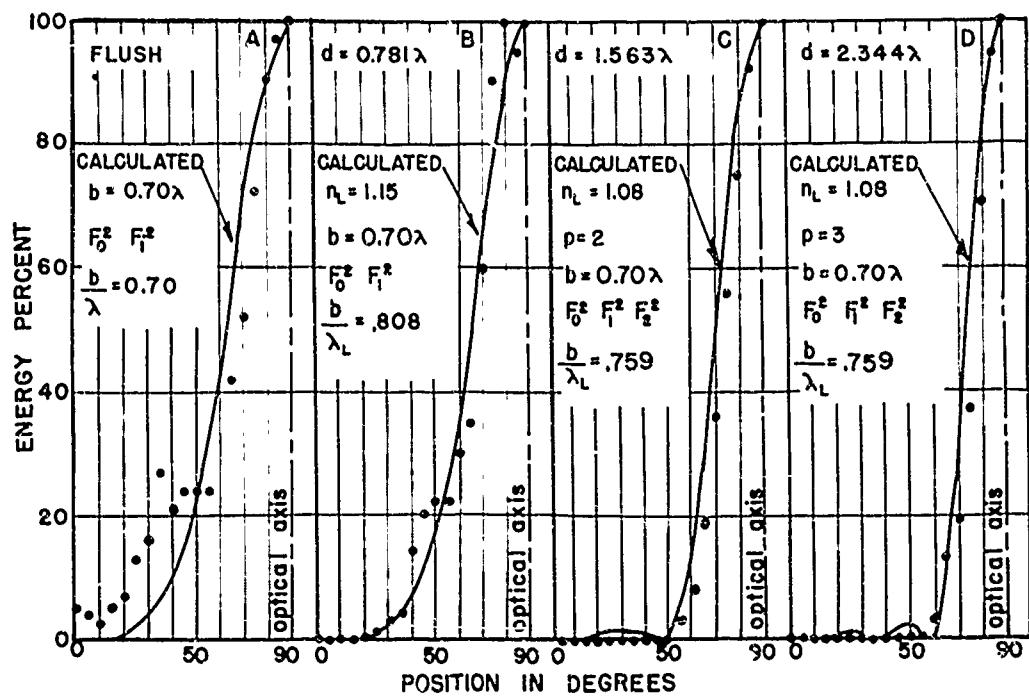

Fig. 12

NARROW RECTANGULAR F1114 LENS
H PATTERNS

$$F_0{}^2 = \left(\frac{\cos\frac{\pi}{2}\cos\theta}{\sin\theta}\right)^2 \qquad F_1{}^2 = \left(\frac{2J_1\frac{\pi a}{\lambda_L}\cos\theta}{\frac{\pi a}{\lambda_L}\cos\theta}\right)^2 \qquad F_2{}^2 = \left(\frac{\sin\frac{p\pi}{n_L}\sin^2\theta}{\sin\frac{\pi}{n_L}\sin^2\theta}\right)^2$$

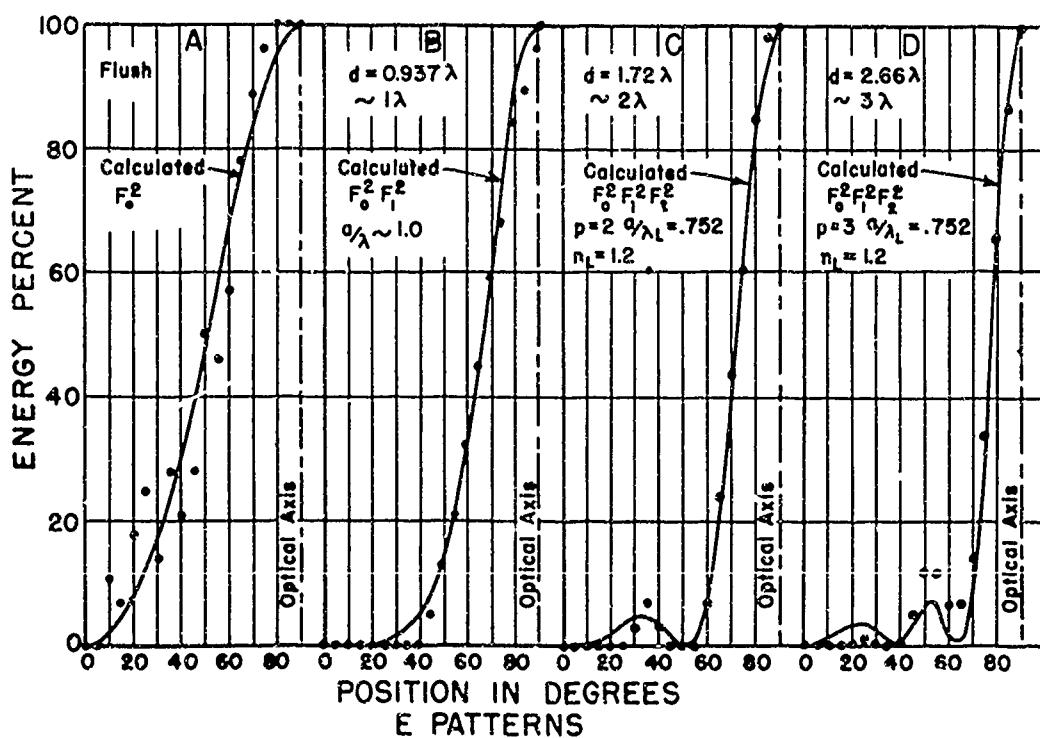### POSITION IN DEGREES
E PATTERNS

Fig. 13

CIRCULAR F1114 LENS – FLAT FACED AT EACH END

$$E_E^2 = \left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2 \left(\frac{J_1\, \frac{a\pi}{\lambda_L}\cos\theta}{\frac{a\pi}{\lambda}\cos\theta}\right)^2 \left(\frac{\sin\frac{p\pi}{n_L}\sin^2\theta}{\sin\frac{\pi}{n_L}\sin^2\theta}\right)^2 .$$
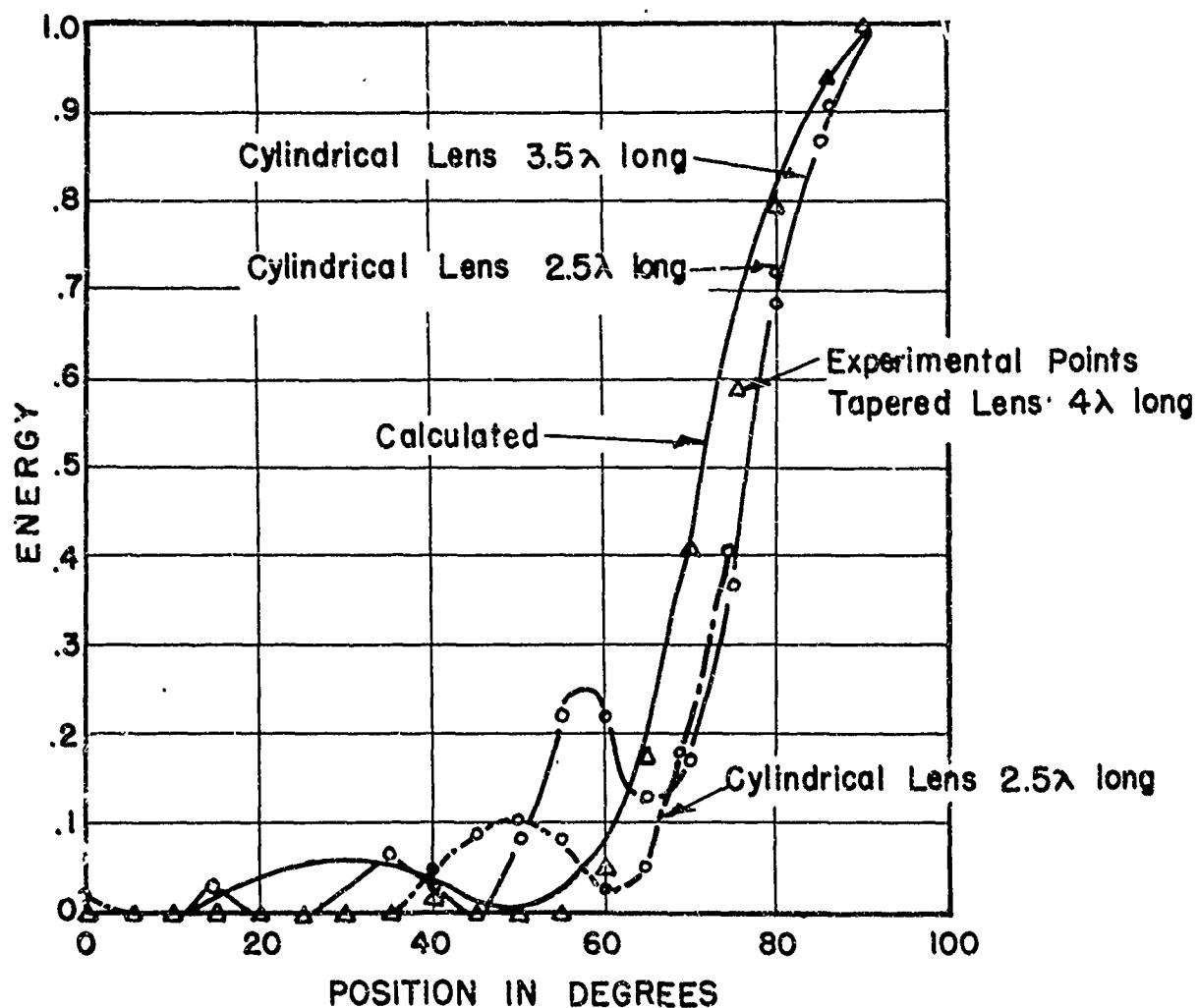
$$n_L = 1.2$$
$$p = 2$$

$$\frac{a}{\lambda_L} = .750$$

Fig. 14

PATTERN OF TAPERED LENS
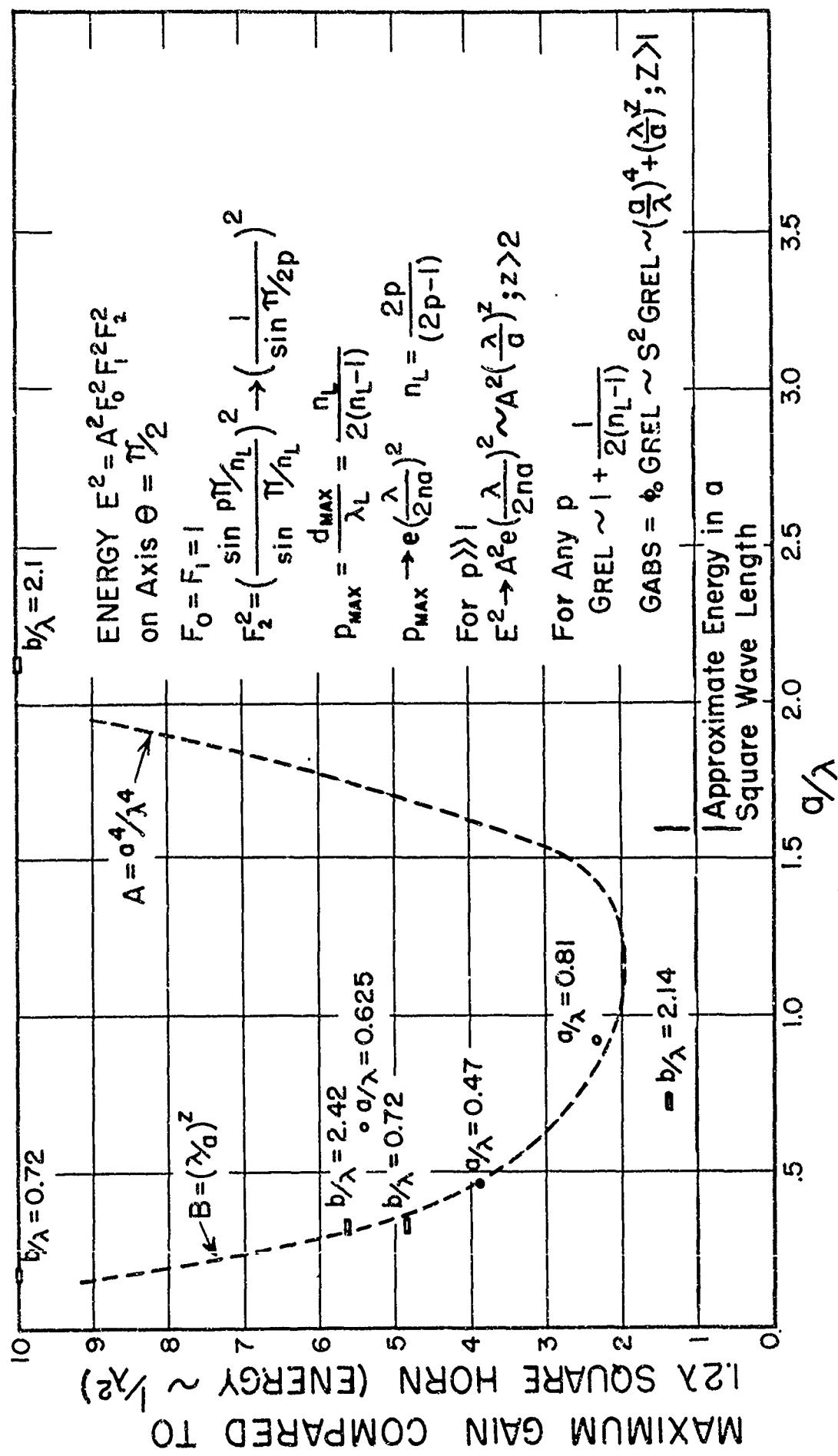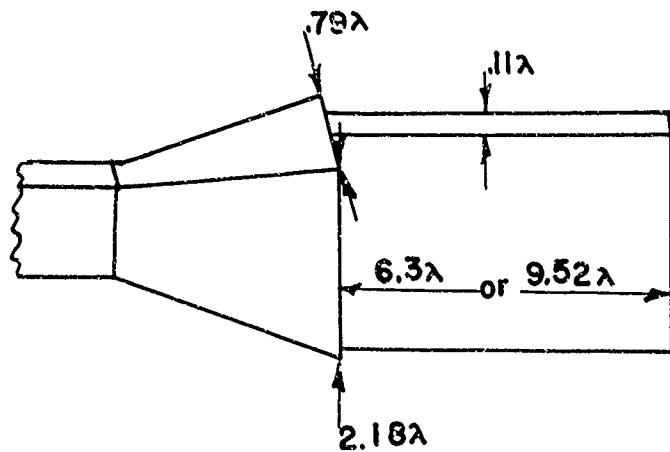COMPARED TO CYLINDRICAL LENSES

BASE DIAM. .625λ F1114

ABSOLUTE GAIN OF LENSES

Fig. 15

ENERGY $E^2 = A^2 F_0^2 F_1^2 F_2^2$
on Axis $\theta = \pi/2$

$F_0 = F_1 = 1$

$F_2^2 = \left(\dfrac{\sin p\pi/n_L}{\sin \pi/n_L}\right)^2 \to \left(\dfrac{1}{\sin \pi/2p}\right)^2$

$p_{MAX} = \dfrac{d_{MAX}}{\lambda_L} = \dfrac{n_L}{2(n_L-1)}$

$p_{MAX} \to e\left(\dfrac{\lambda}{2na}\right)^2 \qquad n_L = \dfrac{2p}{(2p-1)}$

For $p \gg 1$

$E^2 \to A^2 e\left(\dfrac{\lambda}{2na}\right)^2 \sim A^2\left(\dfrac{\lambda}{2na}\right)^2$

For Any p

$GREL \sim 1 + \dfrac{1}{2(n_L-1)}$

$GABS = \oint_0 GREL \sim S^2\, GREL \sim \left(\dfrac{a}{\lambda}\right)^4 + \left(\dfrac{z}{a/\lambda}\right)^z ; z>1$

$z > 2$

$b/\lambda = 2.1$

$A = a^4/\lambda^4$

$B = (\lambda/a)^z$

$b/\lambda = 0.72$

$b/\lambda = 2.42$

$a/\lambda = 0.625$

$b/\lambda = 0.72$

$a/\lambda = 0.81$

$a/\lambda = 0.47$

$b/\lambda = 2.14$

Approximate Energy in a Square Wave Length

Horizontal axis: a/λ

Vertical axis: 1.2λ SQUARE HORN (ENERGY ~ $1/\lambda^2$) MAXIMUM GAIN COMPARED TO

.79λ

.11λ

6.3λ or 9.52λ

2.18λ

E PLANE .79λ X 2.18λ HORN

LUCITE LENS
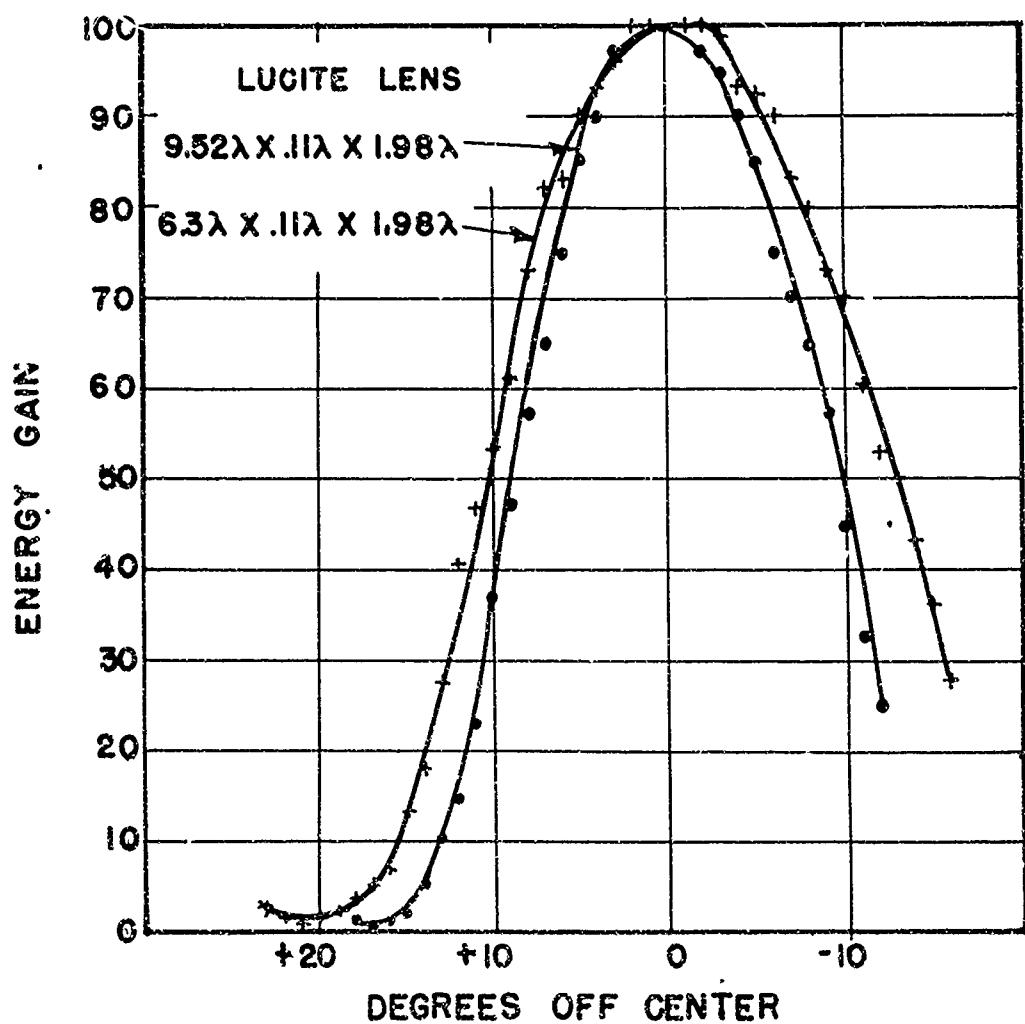
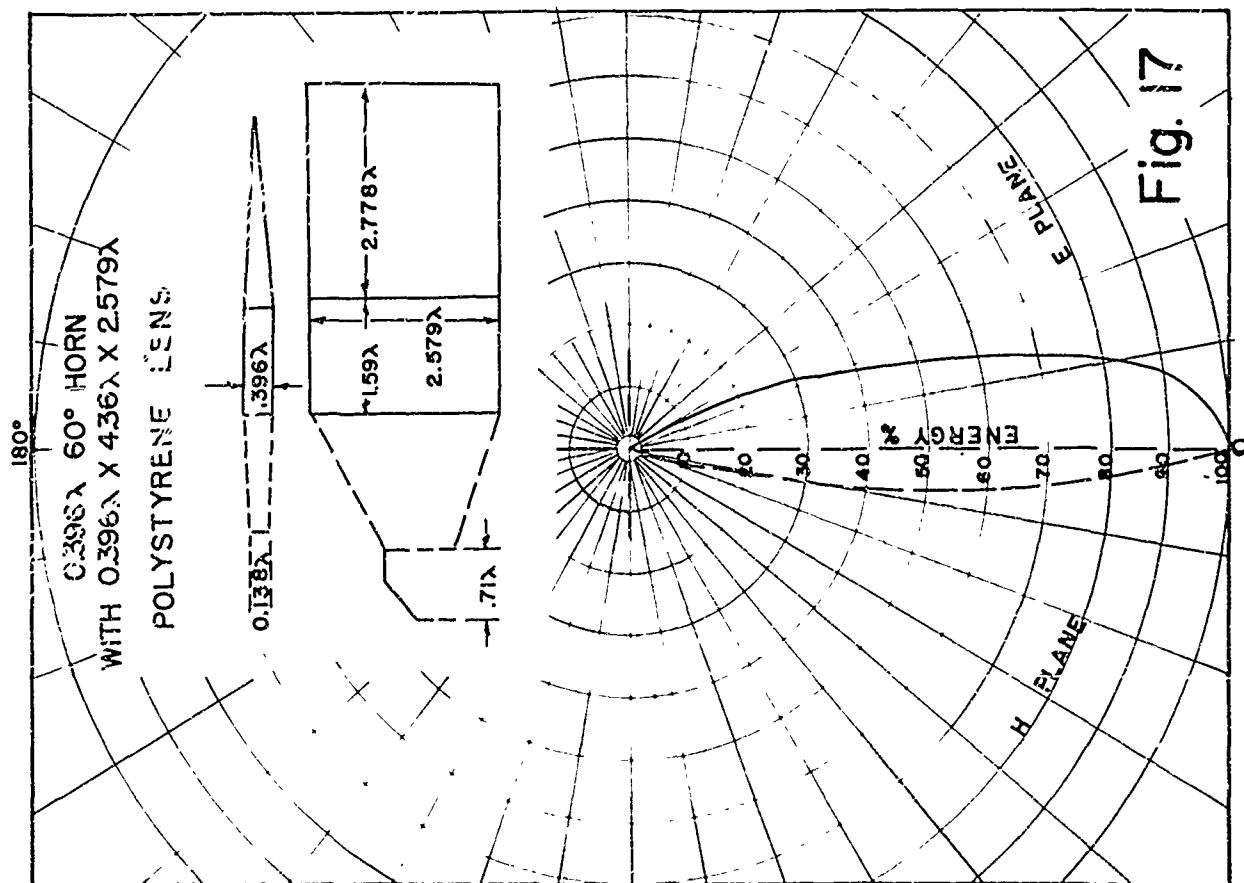9.52λ X .11λ X 1.98λ

6.3λ X .11λ X 1.98λ

ENERGY GAIN
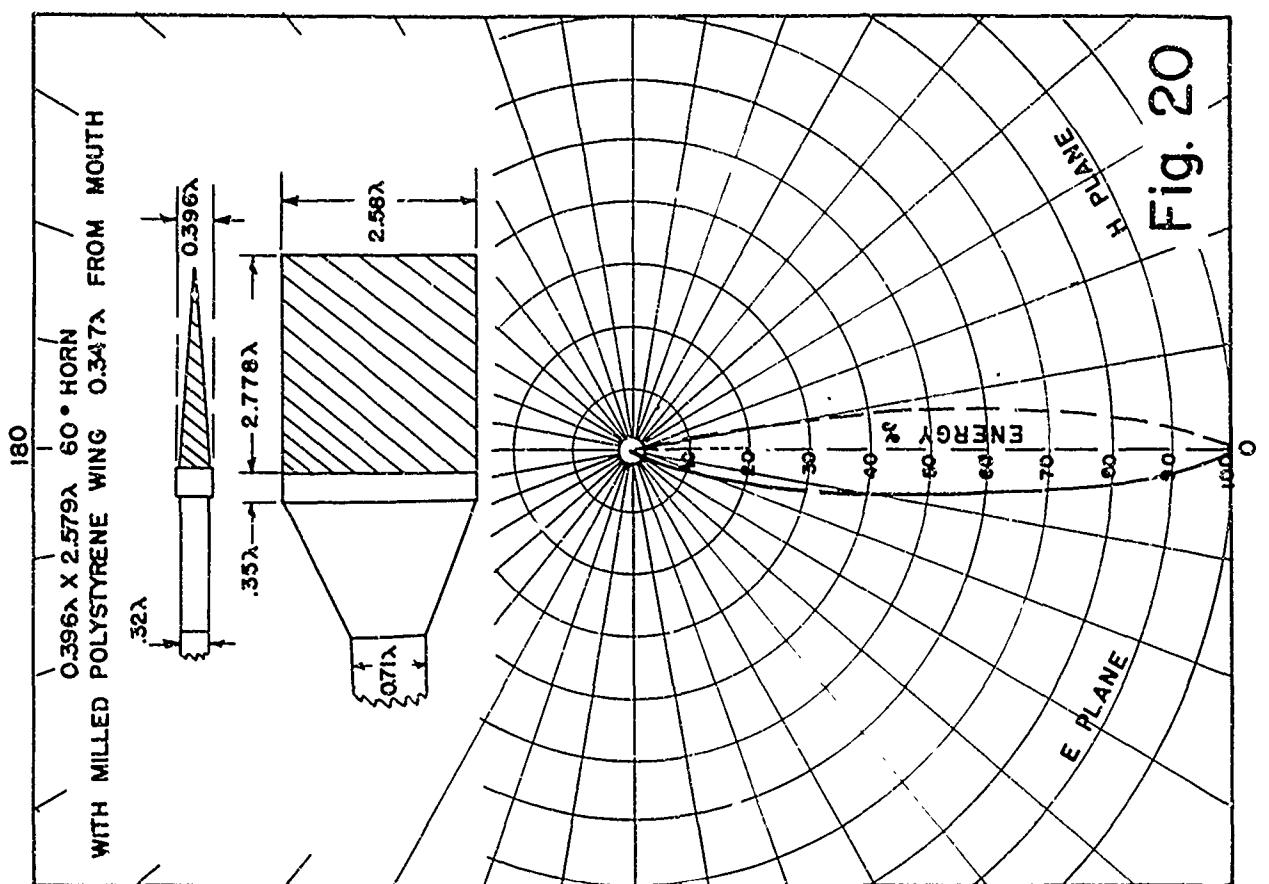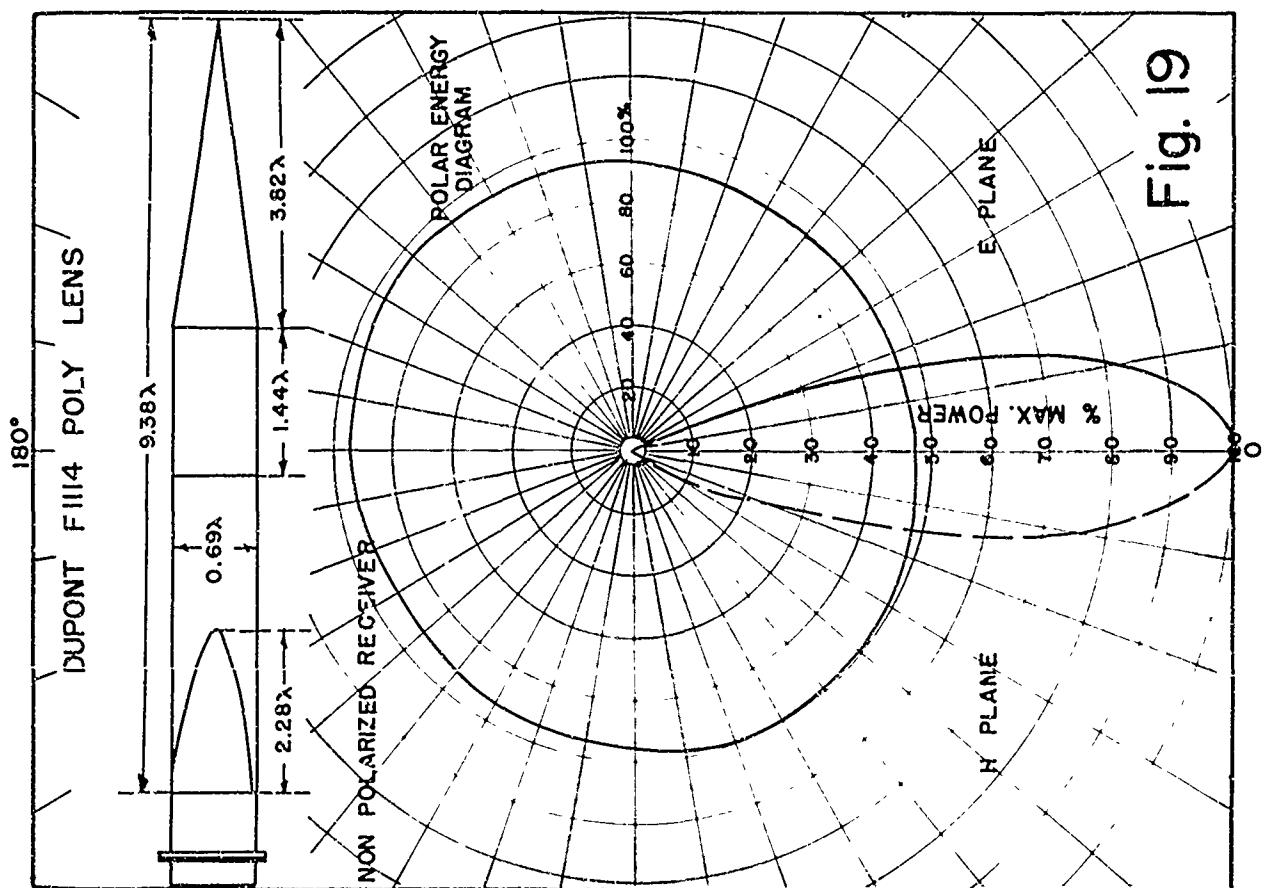
DEGREES OFF CENTER

Fig. 16

VARIATION OF MAIN BEAM WIDTH
WITH LENS LENGTH

Fig. 18

40° HORN
WITH GROUND PLATE GLASS LENS

180°

0.396λ
2.579λ
2.78λ
9.40λ

H PLANE
E PLANE
% ENERGY

Fig. 17

0.396λ 60° HORN
WITH 0.396λ X 4.36λ X 2.579λ
POLYSTYRENE LENS

180°

1.396λ
1.59λ
2.778λ
2.579λ
0.138λ
.71λ

E PLANE
H PLANE
% ENERGY

0.396λ X 2.579λ 60° HORN
WITH MILLED POLYSTYRENE WING 0.347λ FROM MOUTH

0.396λ

2.58λ

2.778λ

.35λ

.32λ

0.71λ

180

ENERGY %

H PLANE

E PLANE

Fig. 20

DUPONT F1114 POLY LENS

180°

9.38λ

3.82λ

1.44λ

0.69λ

2.28λ

NON POLARIZED RECEIVER

POLAR ENERGY DIAGRAM

100%

80

60

40

20

% MAX. POWER

E PLANE

H PLANE

Fig. 19

TITLE: Wave Length Lenses

AUTHOR(S): Wilkes, Gilbert
ORIGINATING AGENCY: Johns Hopkins Univ., Applied Physics Lab., Silver Spring, Md.

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ABSTRACT:

The energies considered have lower frequency than that of light, and the wave lengths are such that the lenses have dimensions of the order of a wave length, for the more usual applications. Energy velocity in a lens approaches that of light with a decreasing cross section. The thinner the lens, the longer it can be made so as to increase its exposed area. An increased gain results. Experimental data correlating their different properties is given with tentative supporting theories where possible.

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